SUPERGRAVITY

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Supergravity

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notes by

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DIFFERENTIABLE MANIFOLD

$$x^m \to x'^m = x^m - \xi^m(x)$$

$$f(x) \to f'(x) \qquad f'(x') = f(x)$$

$$f(x - \xi(x)) + \delta f(x) = f(x)$$

$$\delta f(x) = \xi^m \partial_m f(x) \qquad \partial_m \equiv \frac{\partial}{\partial x^m}$$

Covariant vector: $\delta u_m = \xi^\ell \partial_\ell u_m + \partial_m \xi^\ell u_\ell$

Contravariant vector: $\delta v^m = \xi^\ell \partial_\ell v^m - \partial_\ell \xi^m v^\ell$

Density of weight p: $\delta h = \xi^m \partial_m h + p\, \partial_m \xi^m h$

Vector density of weight p: $\delta K^m = \xi^\ell \partial_\ell \xi^m - \partial_\ell \xi^m K^\ell + p\, \partial_\ell \xi^\ell K^m$

Geometric operations:

Gradient of a scalar: $\partial_m f$

Curl of a covariant vector: $\partial_m u_n - \partial_n u_m$

Divergence of a vector density of weight $p = 1$: $\partial_m K^m$

Divergence of an antisymmetric tensor density of weight $p = 1$: $\partial_m K^{mn}$

$$K^{mn} = -K^{nm}$$

AFFINE SPACE, CONNECTION $\Gamma_{mn}{}^{\ell}$

Covariant derivative — Connection coefficients

$$\mathcal{D}_m u_n = \partial_m u_n - \Gamma_{mn}{}^{\ell} u_\ell$$

$$\mathcal{D}_m v^n = \partial_m v^n + \Gamma_{m\ell}{}^{n} v^\ell$$

will transform as tensors if

$$\delta \Gamma_{mn}{}^{\ell} = \xi^s \partial_s \Gamma_{mn}{}^{\ell} + \partial_m \xi^s \Gamma_{sn}{}^{\ell} + \partial_n \xi^s \Gamma_{ms}{}^{\ell}$$

$$- \partial_s \xi^\ell \Gamma_{mn}{}^{s} + \partial_m \partial_n \xi^\ell$$

Except for the last term (symmetric in m, n) this is the Transf. law of a tensor. $\Gamma_{mn}{}^{\ell}$ is not a tensor, but the antisymmetric combination

$$T_{mn}{}^{\ell} = \Gamma_{mn}{}^{\ell} - \Gamma_{nm}{}^{\ell}$$

is a tensor, called the TORSION.

Covariant derivatives do not commute:

$$[\mathcal{D}_m, \mathcal{D}_n] v^\ell = R_{mns}{}^{\ell} v_s - T_{mn}{}^{s} \mathcal{D}_s v^\ell$$

where

$$R_{mns}{}^{\ell} = \partial_m \Gamma_{ns}{}^{\ell} - \partial_n \Gamma_{ms}{}^{\ell} - \Gamma_{ms}{}^{r} \Gamma_{nr}{}^{\ell} + \Gamma_{ns}{}^{r} \Gamma_{mr}{}^{\ell}$$

is the curvature tensor

$$R_{mns}{}^{\ell} = -R_{nms}{}^{\ell}$$

If both curvature and torsion vanish, one can find a coordinate frame in which the connection coefficients vanish: the affine space is flat.

JACOBI IDENTITY

$$\oint_{rmn} [\mathcal{D}_r, [\mathcal{D}_m, \mathcal{D}_n]] v^l = 0$$

The two Bianchi identities follow

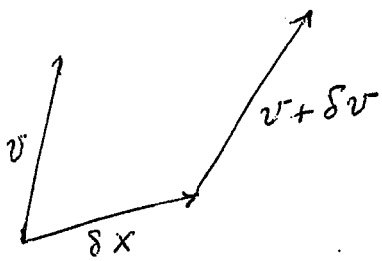
$$\oint_{rmn} \left(\mathcal{D}_r T_{mn}{}^l + T_{rm}{}^s T_{sn}{}^l - R_{rmn}{}^l \right) = 0$$

$$\oint_{rmn} \left(\mathcal{D}_r R_{mns}{}^l + T_{rm}{}^t R_{tns}{}^l \right) = 0$$

(Observe the special cases when the torsion vanishes).

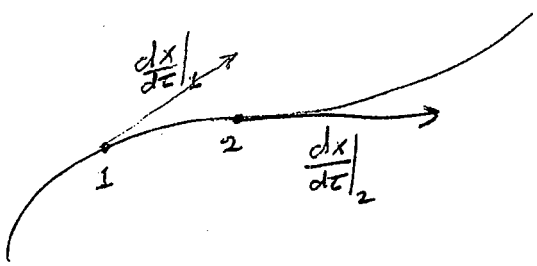
EQ. FOR THE GEODESICS IN AN AFFINE SPACE,

PARALLEL TRANSPORT. OF A <u>VECTOR</u>



$$\delta v^m = -\Gamma_{ne}{}^{m}\,\delta x^n v^e$$

A GEODESIC IS A SELFPARALLEL CURVE



$$\left.\frac{dx^m}{d\tau}\right|_2 = \left.\frac{dx^m}{d\tau}\right|_1 + \delta\,\frac{dx^m}{d\tau}$$

$$\left.\frac{dx^m}{d\tau}\right|_1 + (\tau_2-\tau_1)\left.\frac{d^2x^m}{d\tau^2}\right|_1 = \left.\frac{dx^m}{d\tau}\right|_1 - \Gamma_{ne}{}^{m}\,(\tau_2-\tau_1)\left.\frac{dx^n}{d\tau}\right|_1\left.\frac{dx^e}{d\tau}\right|_1$$

$$\frac{d^2x^m}{d\tau^2} = -\Gamma_{ne}{}^{m}\,\frac{dx^n}{d\tau}\,\frac{dx^e}{d\tau}$$

COVARIANT DERIVATIVE OF A VECTOR <u>FIELD</u>. $v^m(x)$

$$x'^m = x^m + \delta x^m \qquad v^e(x') = v^e(x) + \partial_m v^e\,\delta x^m$$

$$v^e(x') - (v^e(x) + \delta v^e(x)) = v^e(x) + \partial_m v^e\,\delta x^m - v^e(x)$$

$$+\Gamma_{mz}{}^{e}(x)\,\delta x^m v^z(x) = \delta x^m\left(\partial_m v^e + \Gamma_{mz}{}^{e}(x)\,v^z(x)\right)$$

$$\overset{\text{Def.}}{=} \delta x^m\,\mathcal{D}_m v^e$$

MORE STRUCTURE : METRIC SPACE

$$ds^2 = g_{mn}\, dx^m dx^n$$

$$g_{mn}(x) = g_{nm}(x)$$

One can raise and lower indices by using g_{mn} and its inverse matrix g^{mn}

$$g_{mn} g^{nl} = \delta_m{}^l ,$$

$$u_m = g_{mn} v^n , \quad v^n = g^{nm} u_m ,$$

Consistency with covariant differentiation

$$g_{nm}(\mathcal{D}_l v^m) = \mathcal{D}_l (g_{nm} v^m)$$

Requires that the metric tensor be covariantly constant

$$\mathcal{D}_l g_{nm} = 0 ,$$

If an affine space satisfies this and has zero torsion, it is called a Riemann space.

If one assumes that the metric is covariantly constant

$$D_e g_{mn} = \partial_e g_{mn} - \Gamma_{em}{}^s g_{sn} - \Gamma_{en}{}^s g_{ms} = 0$$

one can solve for the connection coefficients in terms of the metric and its derivatives. Let $\Gamma_{em}{}^s g_{sn} = \Gamma_{emn}$

The eq. is

$$\partial_e g_{mn} - \Gamma_{emn} - \Gamma_{enm} = 0$$

Apply a cyclic permutation to the indices.

$$\partial_m g_{ne} - \Gamma_{mne} - \Gamma_{men} = 0$$

$$\partial_n g_{em} - \Gamma_{nem} - \Gamma_{nme} = 0$$

Add the first two and subtract the third

$$\partial_e g_{mn} + \partial_m g_{ne} - \partial_n g_{em} - (\Gamma_{emn} + \Gamma_{men}) - (\Gamma_{enm} - \Gamma_{nem}) - (\Gamma_{mne} - \Gamma_{nme}) = 0$$

Writing $\Gamma_{emn} = \frac{1}{2}(\Delta_{emn} + T_{emn})$

where $\Delta_{emn} = \Delta_{men}$, $T_{emn} = -T_{men} = T_{em}{}^s g_{sn}$

one has

$$\partial_e g_{mn} + \partial_m g_{en} - \partial_n g_{em} - T_{enm} - T_{mne} = \Delta_{emn}$$

so that

$$\Gamma_{emn} = \frac{1}{2}(\partial_e g_{mn} + \partial_m g_{en} - \partial_n g_{em} - T_{enm} - T_{mne} + T_{emn})$$

and $\Gamma_{em}{}^s = \Gamma_{emn} g^{ns}$

If the torsion vanishes, the last three terms are absent and one has the standard formula valid in a Riemann space

In a Riemann space, $T_{mn}{}^{\ell} = 0$, the curvature tensor satisfies, in addition to

$$R_{mn,\ell}{}^{s} = -R_{nm,\ell}{}^{s}$$

the Bianchi identities

$$\oint_{mn\ell} R_{mn,\ell}{}^{s} = 0$$

and

$$\oint_{tmn} \mathcal{D}_t R_{mn,\ell}{}^{s} = 0 .$$

Furthermore, since the tangent space group is the (pseudo-)orthogonal group,

$$R_{mn,\ell s} = -R_{mn,s\ell}$$

Show that, together with the two first eq.s above, this implies

$$R_{mn,\ell s} = R_{\ell s,mn} .$$

Let $D = \det M = e^{\operatorname{tr} \log M}$

Vary M by δM

$$\delta D = D \operatorname{tr} \delta \log M = D \operatorname{tr} M^{-1} \delta M$$

This is correct under the trace, even though M and δM may not commute.

$$\delta \sqrt{g} = \frac{1}{2} g^{-1/2} \delta g = \frac{1}{2} \sqrt{g}\, g^{mn} \delta g_{mn} \qquad g = \det g_{mn}$$

For $\delta g_{mn} = \xi^{\ell} \partial_{\ell} g_{mn} + \partial_m \xi^{\ell} g_{\ell n} + \partial_n \xi^{\ell} g_{m\ell}$

this gives

$$\delta \sqrt{g} = \xi^{\ell} \partial_{\ell} \sqrt{g} + \partial_{\ell} \xi^{\ell} \sqrt{g} = \partial_{\ell} (\xi^{\ell} \sqrt{g})$$

$\sqrt{g}$ is a density of weight $p = 1$.

If v^m is a vector, $\sqrt{g}\, v^m$ is a vector density of weight $p=1$,

The Christoffel formula

$$\Gamma_{em}{}^{s} = \frac{1}{2}\left(\partial_e g_{mn} + \partial_m g_{en} - \partial_n g_{em}\right) g^{ns}$$

gives

$$\Gamma_{em}{}^{m} = \frac{1}{2}\left(\partial_e g_{mn}\right) g^{mn} = \frac{1}{2g}\,\partial_e g = \frac{1}{\sqrt{g}}\,\partial_e \sqrt{g}$$

where $g = \det g_{mn}$

Observe that $\Gamma_{ms}{}^{m} = \frac{1}{\sqrt{g}}\,\partial_s \sqrt{g}$ implies

$$\mathcal{D}_m v^m = \partial_m v^m + \Gamma_m{}^m{}_e v^e = \frac{1}{\sqrt{g}}\,\partial_m\left(\sqrt{g}\, v^m\right)$$

Example: Equation for the geodesics.

$$I = \int_A^B ds = \int_A^B \sqrt{g_{mn} \frac{dx^m}{d\tau} \frac{dx^n}{d\tau}} \, d\tau$$

Vary the path keeping the end-points fixed

$x^m(\tau) \to x^m(\tau) + \delta x^m(\tau)$ $\qquad \delta x^m = 0$ at A, B.

$$\delta I = \frac{1}{2} \int_A^B \frac{1}{\sqrt{}} \left(\partial_\ell g_{mn} \, \delta x^\ell \frac{dx^m}{d\tau} \frac{dx^n}{d\tau} + 2 g_{mn} \frac{dx^m}{d\tau} \frac{d\delta x^n}{d\tau} \right) d\tau$$

$$= \frac{1}{2} \int_A^B \left(\partial_\ell g_{mn} \, \delta x^\ell \frac{dx^m}{ds} \frac{dx^n}{ds} + 2 g_{mn} \frac{dx^m}{ds} \frac{d\delta x^n}{ds} \right) ds$$

$$= \int_A^B \left(\frac{1}{2} \partial_\ell g_{mn} \frac{dx^m}{ds} \frac{dx^n}{ds} - \partial_n g_{m\ell} \frac{dx^m}{ds} \frac{dx^n}{ds} - g_{m\ell} \frac{d^2 x^m}{ds^2} \right) \delta x^\ell \, ds$$

$$= - \int_A^B \left(\frac{d^2 x^m}{ds^2} + \Gamma^m_{nr} \frac{dx^n}{ds} \frac{dx^r}{ds} \right) g_{m\ell} \, \delta x^\ell \, ds$$

$$\frac{d^2 x^m}{ds^2} + \Gamma^m_{nr} \frac{dx^n}{ds} \frac{dx^r}{ds} = 0 .$$

Example: Maxwell's eq.s in curved space

$$F_{mn} = \partial_m A_n - \partial_n A_m = -F_{nm}$$

$$I = -\frac{1}{4}\int F_{mn} F^{mn} \sqrt{g}\, d^4x$$

$$\delta I = \int \delta A_n \, \partial_m (\sqrt{g}\, F^{mn})\, d^4x \qquad \text{keep } g_{mn} \text{ fixed.}$$

$$\begin{cases} \partial_m (\sqrt{g}\, F^{mn}) = 0 \\ \partial_e F_{mn} + \partial_m F_{ne} + \partial_n F_{em} = 0 \end{cases}$$

$$\delta(\sqrt{g}\,R) = \sqrt{g}\,(R_{mn}\,\delta g^{mn}) + \sqrt{g}\;\delta R_{mn}\,g^{mn} + \delta\sqrt{g}\;R$$

$$\delta\, R_{mne}{}^{s} = \mathcal{D}_m(\delta\Gamma_{ne}{}^{s}) - \mathcal{D}_n(\delta\Gamma_{me}{}^{s})$$

$$R = R_{mn}{}^{mn} = R_{mne}{}^{n}\,g^{me} = R_{me}\,g^{me}$$

$$R_{me} = R_{mne}{}^{n}$$

$$\delta R_{mn} = \mathcal{D}_m(\delta\Gamma_{en}{}^{e}) - \mathcal{D}_e(\delta\Gamma_{mn}{}^{e})$$

$$\sqrt{g}\,g^{mn}\delta R_{mn} = \sqrt{g}\left[\mathcal{D}_m(g^{mn}\delta\Gamma_{en}{}^{e}) - \mathcal{D}_e(g^{mn}\delta\Gamma_{mn}{}^{e})\right]$$

$$= \partial_m(\sqrt{g}\,g^{mn}\delta\Gamma_{en}{}^{e}) - \partial_e(\sqrt{g}\,g^{mn}\delta\Gamma_{mn}{}^{e})$$

Integrates to zero. It remains

$$\int\delta(\sqrt{g}\,R) = \int\sqrt{g}\;(R_{mn} - \tfrac{1}{2}g_{mn}R)\;\delta g^{mn}\,d^4x$$

which gives Einstein's eq.s

$$R_{mn} - \tfrac{1}{2}g_{mn}R = 0$$

To transform $\sqrt{g}\,g^{mn}\delta R_{mn}$ to a total derivative we have used the relation between g_{mn} and $\Gamma_{mn}{}^{e}$, or $\mathcal{D}_e g_{mn} = 0$. If we had not, the equations obtained by taking independent variations $\delta\Gamma_{mn}{}^{e}$ would clearly give the relation between connection and metric. So, assuming that the Γ are symmetric we would have a Riemann space as a consequence of the dynamics.

LOCAL BASIS VECTORS (AFFINE SPACE)

$e_a(x)$ They have components (curvilinear) $e_a{}^m(x)$. Their duals $e^a(x)$ have curvilinear components $e_m{}^a(x)$. Duality is expressed by

$$e_a{}^m e_m{}^b = \delta_a{}^b \qquad \text{(they are inverse matrices)}$$

Vierbein field in fourdims. Vielbein in general.

$$v^m = v^a e_a{}^m \qquad v^a = v^m e_m{}^a$$

$$u_m = e_m{}^a u_a \qquad u_a = e_a{}^m u_m$$

An infinitesimal change of local tangent space frame changes the components as

$$\delta v^a = v^b X_b{}^a \qquad \delta u_a = -X_a{}^b u_b$$

$$v^a u_a \text{ is invariant}$$

Covariant derivatives

$$\mathcal{D}_m v^a = \partial_m v^a + v^b \omega_{mb}{}^a$$

$$\mathcal{D}_m u_a = \partial_m u_a - \omega_{ma}{}^b u_b$$

$\omega_{ma}{}^b$ "rotation coefficients"

The rotation coeff.s must transform as

$$\delta \omega_{ma}{}^{b} = \omega_{ma}{}^{c} X_{c}{}^{b} - X_{a}{}^{c} \omega_{mc}{}^{b} - \partial_m X_a{}^{b}$$

So that the covariant derivatives transform correctly. Under general coord. transf.s the tangent space vectors behave as scalars and the $\omega_{ma}{}^{b}$ as covariant vectors with index m.

One can also consider quantities with both tangent space indices and curvilinear indices. The vielbein is such a quantity. The covariant derivative is then constructed with both Γ and ω, e.g.

$$\mathcal{D}_m e_n{}^{a} \overset{\text{Def.}}{=} \partial_m e_n{}^{a} - \Gamma_{mn}{}^{l} e_l{}^{a} + e_n{}^{b} \omega_{mb}{}^{a}$$

For consistency of $\mathcal{D}_m v^a$, $\mathcal{D}_m v^l$ and $v^l = v^a e_a{}^l$ etc. one must require the covariant der. of the vielbein to vanish

$$\mathcal{D}_m e_n{}^{a} = 0 .$$

This gives the relation

$$\Gamma_{mn}{}^{l} e_l{}^{a} = \partial_m e_n{}^{a} + e_n{}^{b} \omega_{mb}{}^{a} \overset{\text{Def}}{=} \tilde{\mathcal{D}}_m e_n{}^{a}$$

By def. a tilde covariant derivative "covariantizes" only with respect to tangent space indices.

One obtains for the Torsion

$$T_{mn}{}^{a} = T_{mn}{}^{l} e_{l}{}^{a} = \tilde{\mathcal{D}}_{m} e_{n}^{a} - \tilde{\mathcal{D}}_{n} e_{m}{}^{a}$$

The curl-like combination on the right hand side is covariant!

The commutator of two tilde derivatives gives

$$[\tilde{\mathcal{D}}_{m}, \tilde{\mathcal{D}}_{n}] v^{a} = R_{mnb}{}^{a} v^{b}$$

where

$$R_{mna}{}^{b} = \partial_{m} \omega_{na}{}^{b} - \partial_{n} \omega_{ma}{}^{b} - \omega_{ma}{}^{c} \omega_{nc}{}^{b} + \omega_{na}{}^{c} \omega_{mc}{}^{b}$$

is a world tensor with respect to m, n and a tangent space tensor with respect to a, b.

The commutator of two fully covariant derivatives has an extra term

$$[\mathcal{D}_{m}, \mathcal{D}_{n}] v^{a} = R_{mnb}{}^{a} v^{b} - T_{mn}{}^{l} \mathcal{D}_{l} v^{a}$$

Comparing with an earlier formula

$$[\mathcal{D}_{m}, \mathcal{D}_{n}] v^{s} = R_{mn,l}{}^{s} v^{l} - T_{mn}{}^{l} \mathcal{D}_{l} v^{s}$$

one sees, since the vielbein is covariantly constant, that

$$R_{mn,a}{}^{b} = R_{mn,l}{}^{s} e_{a}{}^{l} e_{s}{}^{b}$$

LOCAL BASIS VECTORS (METRIC SPACE).

Here there exist η_{ab}, η^{ab} which can be used to raise and lower tangent space indices. They are numerically invariant, so

$$X_{ab} \overset{\text{Def}}{=} X_a{}^c \eta_{cb} = - X_{ba} .$$

$\omega_{m,a}{}^b$ and $R_{mn,a}{}^b$, as matrices in ab belong to the algebra of the (pseudo-) orthogonal group:

$$\omega_{mab} = \omega_{m\,a}{}^c \eta_{cb} = - \omega_{m\,ba}$$

$$R_{mn,ab} = - R_{mn,ba} \quad (\text{which implies } R_{mn,es} = - R_{mn,se})$$

The metric tensor

$$g_{mn} = e_m{}^a e_n{}^b \eta_{ab}$$

is covariantly conserved, since $e_m{}^a$ is.

In a Riemann space the structure group is (pseudo-) orthogonal: there exists the numerically invariant tensor η_{ab}. In addition the Torsion vanishes

$$T_{mn}{}^{a} = \tilde{\partial}_m e_n{}^a - \tilde{\partial}_n e_m{}^a$$

$$= \partial_m e_n{}^a - \partial_n e_m{}^a + e_n{}^b \omega_{mb}{}^a - e_m{}^b \omega_{nb}{}^a = 0$$

These eq.s can be solved for ω in terms of the vielbein and its derivatives. Define

$$\omega_{cab} = e_c{}^m \omega_{mab} = -\omega_{cba}$$

$$C_{bca} = (\partial_m e_{na} - \partial_n e_{ma}) e_b{}^m e_c{}^n = -C_{cba}$$

The equations are

$$C_{bca} + \cancel{\omega_{bca}} - \omega_{cba} = 0$$

Apply a cyclic permutation to the indices

$$C_{cab} + \omega_{cab} - \cancel{\omega_{acb}} = 0$$

$$C_{abc} + \cancel{\omega_{abc}} - \cancel{\omega_{bac}} = 0$$

Add the first two and subtract the third:

$$\omega_{cab} = -\frac{1}{2}\left(C_{cab} - C_{abc} + C_{bca}\right)$$

Then

$$\omega_{mab} = e_m{}^c \omega_{cab}.$$

$e = \det e_m{}^a = \sqrt{\det g_{mn}}$ is a density of weight $p = 1$.

$$\delta e = e \; e_a{}^m \delta e_m{}^a$$

For $\delta e_m{}^a = \xi^\ell \partial_\ell e_m{}^a + \partial_m \xi^\ell e_\ell{}^a$ one finds

$$\delta e = \xi^\ell \partial_\ell e + e \; e_a{}^m (\partial_m \xi^\ell) e_\ell{}^a$$

$$= \xi^\ell \partial_\ell e + \partial_\ell \xi^\ell e = \partial_\ell (\xi^\ell e) .$$

Clearly e is invariant under local Lorentz transf.s: for $\delta e_m{}^a = e_m{}^b X_b{}^a$,

$$\delta e = e \; e_a{}^m e_m{}^b X_b{}^a = e \; X_a{}^a = 0$$

Actually, this would vanish if the tangent space group consisted of all linear transf.s of $\det = 1$, since the inf. matrix $X_a{}^b$ is then traceless.

A USEFUL IDENTITY, VALID IN A SPACE WITH AFFINE CONNECTION (NO METRIC IS NECESSARY) Used in partial integration.

$$\partial_m (e\, v^a e_a{}^m) = e\left(\mathcal{D}_a v^a + v^b T_{ba}{}^a\right)$$

Check! (assume $\omega_{ma}{}^a = 0$) $\qquad \mathcal{D}_a = e_a{}^m \mathcal{D}_m$

We can use it to calculate $\delta(eR)$.

First observe that

$$\delta R_{mn,a}{}^b = \tilde{\mathcal{D}}_m(\delta\omega_{n\,a}{}^b) - \tilde{\mathcal{D}}_n(\delta\omega_{m\,a}{}^b)$$

$$R = R_{mn,a}{}^b e^{am} e_b{}^n$$

$$\delta R = \left[\tilde{\mathcal{D}}_m(\delta\omega_{na}{}^b) - \tilde{\mathcal{D}}_n(\delta\omega_{m\,a}{}^b)\right] e^{am} e_b{}^n$$

$$+2R_n{}^b\, \delta e_b{}^n \qquad \text{where} \quad R_n{}^b = R_{mn,a}{}^b e^{am}$$

Now, using the above identity,

$$\partial_m (e\, \delta\omega_{na}{}^b e^{am} e_b{}^n) = e\, \tilde{\mathcal{D}}_m(\delta\omega_n{}^{ab} e_b{}^n)\, e_a{}^m$$

$$+ e\, \delta\omega_n{}^{cb} e_b{}^n T_{ca}{}^a = e\, \tilde{\mathcal{D}}_m(\delta\omega_n{}^{ab})\, e_b{}^n e_a{}^m$$

$$+ e\, \underbrace{\delta\omega_n{}^{ab}(\tilde{\mathcal{D}}_m e_b{}^n)\, e_a{}^m} + e\, \delta\omega_n{}^{ab} e_b{}^n T_{ca}{}^a$$

$$\hookrightarrow \delta\omega_n{}^{ab}\left(-e_b{}^e \tilde{\mathcal{D}}_m e_e{}^c\, e_c{}^n e_a{}^m\right) = -\frac{1}{2}\delta\omega_n{}^{ab} e_c{}^n T_{ab}{}^c$$

using $\quad \omega_{n\,ab} = -\omega_{n\,ba}$.

Therefore

$$\delta(eR) = e\left(2R_m{}^b - R\, e_m{}^b\right)\delta e_b{}^m$$

$$+ e\,\delta\omega_m{}^{ab}\, e^{cm}\left(T_{abc} - T_{ad}{}^d\,\eta_{bc} + T_{bd}{}^d\,\eta_{ac}\right)$$

$$+ 2\,\partial_m\left(e\,\delta\omega_n{}^{ab}\, e_a{}^m\, e_b{}^n\right)$$

In the action the last term integrates to zero

Therefore

$$\delta\int eR\, d^4x = 0$$

gives, by variation of $e_b{}^m$, the Einstein eq.s

$$R_m{}^b - \frac{1}{2}\, e_m{}^b R = 0$$

and, by variation of $\omega_m{}^{ab} = -\omega_m{}^{ba}$, the vanishing of the torsion

$$T_{ab}{}^c = 0$$

SPINORS. When the tangent space group is (pseudo-)orthogonal one can introduce spinors by referring them to the local orthonormal tangent frame. Use the numerical (x-independent) Dirac matrices

$$\gamma_a \gamma_b + \gamma_b \gamma_a = 2\eta_{ab}$$

and the matrices

$$\Sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$$

The covariant derivative of a spinor is

$$\mathcal{D}_m \psi = \left(\partial_m - \frac{1}{2}\omega_{mab}\Sigma^{ab}\right)\psi .$$

ψ transforms like a spinor under local Lorentz transf.s and like a scalar under general coordinate transf.s. The Lagr. for a Majorana spinor is

$$-\frac{i}{2} e\left(\bar{\psi}\, \gamma^m \mathcal{D}_m \psi + M\bar{\psi}\psi\right) \quad \Bigg| \quad \begin{array}{l} e = \det e_m{}^a \\ \gamma^m = \gamma^a e_a{}^m \end{array}$$

The action for a Majorana field is

$$I_{Maj} = -\frac{i}{2}\int \bar{\psi}(\gamma^m \mathcal{D}_m + M)\psi \; e \; d^4x .$$

No need to symmetrize the derivative since

$$\mathcal{D}_m \bar{\psi}\gamma^m \psi = -\bar{\psi}\gamma^m \mathcal{D}_m \psi ,$$ (For a Dirac spinor one must symmetrize)

Vary ψ:

$$\delta I_{Maj} = -\frac{i}{2}\int \left[\delta\bar{\psi}(\gamma^m \mathcal{D}_m + M)\psi \, e + \bar{\psi}(\gamma^m \mathcal{D}_m + M)\delta\psi \, e\right] d^4x$$

To integrate by parts, use the identity of p. 19

In this case

$$\partial_m(e\,\bar{\psi}\gamma^m\delta\psi) = e\,\mathcal{D}_a(\bar{\psi}\gamma^a\delta\psi) + e\,\bar{\psi}\gamma^b\delta\psi\, T_{ba}{}^a$$

$$= e\,\mathcal{D}_a\bar{\psi}\gamma^a\delta\psi + e\,\bar{\psi}\gamma^a\mathcal{D}_a\delta\psi + e\bar{\psi}\gamma^b\delta\psi\, T_{ba}{}^a$$

Therefore

$$\delta I_{Maj} = -i\int \delta\bar{\psi}(\gamma^m \mathcal{D}_m + M + \gamma^b T_{ba}{}^a)\psi \; e \; d^4x$$

The eq. of motion is

$$\left[\gamma^b(\mathcal{D}_b + T_{ba}{}^a) + M\right]\psi = 0$$

Observe that

$$\gamma^m D_m = \gamma^m \partial_m - \frac{1}{2}\gamma^m \omega_{mab} \Sigma^{ab}$$

For a Majorana spinor

$$\bar{\psi}\gamma^c \Sigma^{ab}\psi = \frac{1}{2}\bar{\psi}\{\gamma^c, \Sigma^{ab}\}\psi$$

$$= \frac{1}{2}\varepsilon^{cabd}\bar{\psi}\gamma_5\gamma_d\psi \qquad \varepsilon_{0123} = -\varepsilon^{0123} = 1$$

$$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$$

Therefore the connection enters the Majorana action only through the combination (axial vector)

$$e_c{}^m \omega_{mab} \varepsilon^{cabd}$$

The same is true for the Dirac action provided the kinetic term is correctly symmetrized

$$L_{Dirac} = -i\left[\frac{1}{2}\bar{\varphi}\gamma^m D_m\varphi - \frac{1}{2}\overline{D_m\varphi}\gamma^m\varphi + M\bar{\varphi}\varphi\right]e$$

(here φ is a complex spinor, $\bar{\varphi} = \varphi^*\gamma^0$)

This fact has the consequence that the action for a spin $\frac{1}{2}$ field is conformally invariant for $M=0$.

If one adds to the Einstein action the action for a Majorana spinor

$$I = -\frac{1}{2}\int e\, R\, d^4x - \frac{i}{2}\int \bar{\psi}(\gamma^m \mathcal{D}_m + M)\psi\, e\, d^4x$$

and varies only $\omega_{m\,ab}$, considered as an independent variable, one finds

$$\delta I = -\frac{1}{2}\int e\, \delta\omega_n{}^{ab} e^{cn}\left(T_{abc} - T_{ad}{}^{d}\eta_{bc} + T_{bd}{}^{d}\eta_{ac}\right)$$

$$+\frac{i}{8}\int e\, \delta\omega_n{}^{ab} e^{cn}\, \varepsilon_{cabd}\, \bar{\psi}\gamma_5\gamma^d\psi$$

Therefore one finds the equation of motion

$$T_{abc} - T_{ad}{}^{d}\eta_{bc} + T_{bd}{}^{d}\eta_{ac} = \frac{i}{4}\varepsilon_{cabd}\,\bar{\psi}\gamma_5\gamma^d\psi$$

Tracing over $b=c$, this gives $T_{ad}{}^{d} = 0$, so

$$T_{abc} = \frac{i}{4}\varepsilon_{abcd}\,\bar{\psi}\gamma_5\gamma^d\psi$$

the torsion does not vanish, it is expressed bilinearly in terms of the spinor field. For this system it is antisymmetric in all three indices

If one varies the Einstein - Majorana action of p. 24 with respect to the vierbein $e_a{}^m$, one finds

$$\delta I = -\int e\left(R_m{}^a - \frac{1}{2} R\, e_m{}^a\right)\delta e_a{}^m\, d^4x$$

$$-\frac{i}{2}\int e\left[\bar\psi\gamma^a\mathcal{D}_m\psi - \bar\psi(\gamma^m\mathcal{D}_m + M)\psi\, e_m{}^a\right]\delta e_a{}^m\, d^4x$$

Therefore one find the eq. of motion

$$R_m{}^a - \frac{1}{2} R\, e_m{}^a = \mathcal{T}_m{}^a \qquad \text{with}$$

$$\mathcal{T}_m{}^a = -\frac{i}{2}\bar\psi\gamma^a\mathcal{D}_m\psi + \frac{i}{2} e_m{}^a \underbrace{\bar\psi(\gamma^m\mathcal{D}_m + M)\psi}_{\text{vanishes by eq. for } \psi \text{ below.}}$$

(energy momentum tensor)

This, together with the previous eq.s

$$T_{abc} = \frac{i}{4}\varepsilon_{abcd}\,\bar\psi\gamma_5\gamma^d\psi$$

and

but this vanishes

$$\gamma^a\left[\left(\mathcal{D}_a + T_{ab}{}^b\right) + M\right]\psi = 0\,,$$

gives the full set of eq.s of motion.

The unknown functions are ψ, $e_m{}^a$ and $\omega_{m\,ab}$

Torsion and curvature are defined in terms of them.

When there is torsion the calculation of p. 17 is modified as follows. One has

$$C_{bca} + \cancel{\omega_{bca}} - \omega_{cba} = T_{bca} \qquad +$$

$$C_{cab} + \omega_{cab} - \cancel{\omega_{acb}} = T_{cab} \qquad +$$

$$C_{abc} + \cancel{\omega_{abc}} - \cancel{\omega_{bac}} = T_{abc} \qquad -$$

So $\qquad \omega_{mab} = e_m{}^c \omega_{cab}$

$$\omega_{cab} = \omega_{cab}(e) + \frac{1}{2}\left(T_{cab} - T_{abc} + T_{bca}\right)$$

where $\omega_{cab}(e) = -\frac{1}{2}\left(C_{cab} - C_{abc} + C_{bca}\right)$

$$C_{bca} = \left(\partial_m e_{na} - \partial_n e_{ma}\right) e_b{}^m e_c{}^n$$

is the old expression for the connection in absence of torsion.

The combination $T_{cab} - T_{abc} + T_{bca}$ is sometimes called the "contorsion".

For the case of the Einstein-Majorana system it simplifies (see p. 24) and one has

$$\omega_{cab} = \omega_{cab}(e) + \frac{i}{8}\,\varepsilon_{cabd}\,\bar{\psi}\gamma_5\gamma^d\psi$$

Clearly, if one substitutes this in the eq.s of motion, one can reinterpret them, take as connection simply $\omega_{cab}(e)$ and consider all additional

terms as self interactions of the spinor field. The same can be done directly in the action. In this sense the same physics lends itself to different geometric interpretations.

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Supplement to notes 13 – 27

p. 14 $\delta(\omega_{ma}{}^b) = \ldots$ is chosen so that

$$\delta(D_m v^a) = X_b{}^a (D_m v^b)$$

under local lorentz transformations. Note that the expression for $\delta\omega_m$ is just

$$\delta\omega_m = -\tilde{D}_m X \quad (= -\partial_m X + [\omega_m, X])$$

p. 19 To understand the formula

$$\delta(R_{mn,a}{}^b) = \ldots$$

make use of the Yang-Mills analogy:

$\omega_{ma}{}^b \longrightarrow \omega_m$ (matrix)

$$\tilde{D}_m = \partial_m - \omega_m$$

$$R_{mn} = \partial_m \omega_n - \partial_n \omega_m - [\omega_m, \omega_n]$$

infinitesimal gauge transformation:

$$\delta\omega_m = -\tilde{D}_m X, \quad \delta R_{mn} = -[X, R_{mn}]$$

(cont.)

Note : $R_{mn} = \tilde{D}_m \omega_n - \tilde{D}_n \omega_m + [\omega_m, \omega_n]$

(cancels one of the commutators)

then $\delta R_{mn} = \tilde{D}_m(\delta\omega_n) - \tilde{D}_n(\delta\omega_m) +$

$\cancel{(\delta \tilde{D}_m)\omega_n} - \cancel{(\delta \tilde{D}_n)\omega_m} + \cancel{[\delta\omega_m, \omega_n]}$

$+ \cancel{[\omega_m, \delta\omega_n]}$

$= \tilde{D}_m(\delta\omega_n) - \tilde{D}_n(\delta\omega_m)$

$= -[\tilde{D}_m, \tilde{D}_n] X = -[X, R_{mn}]$

p. 19 the step: $\tilde{D}_m e_b{}^n = -e_b{}^\ell (\tilde{D}_m e_\ell{}^c) e_c{}^n$

follows from $\tilde{D}_m (e_b{}^n e_n{}^c) = 0$

p. 21 Once the transformation rule (local lorentz) for vectors is given,

$$\delta v^a = X_b{}^a v^b$$

then $\delta\omega_m = -\tilde{D}_m X$

and $\delta\psi = c(X_{ab}\Sigma^{ab})\psi$

c = constant

the constant c is determined by requiring the covariant derivative,

$$\tilde{D}_m \psi = (\partial_m + c\,\omega_{mab}\Sigma^{ab})\psi$$

to transform correctly. Straight forward algebra that uses the relation,

$$[\Sigma^{ab}, \Sigma^{cd}] = \eta^{ad}\Sigma^{bc} + \eta^{bc}\Sigma^{ad} - \eta^{ac}\Sigma^{bd} - \eta^{bd}\Sigma^{ac}$$

(these follow from $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$)

gives the equation $c + 2c^2 = 0$ so $\underline{c = -\frac{1}{2}}$

p. 22 In the majorana rep. the gamma matrices are real and have the property:

γ^0 = antisymmetric

γ^i = symmetric

Thus for <u>any</u> two majorana spinors ψ, φ:

$$\bar{\psi}\gamma^0\varphi = -\psi^T\varphi = +\varphi^T\psi = -\bar{\varphi}\gamma^0\psi$$

$$\bar{\psi}\gamma^i\varphi = \psi^T\gamma^0\gamma^i\varphi = -\varphi^T\gamma^i(-\gamma^0)\psi =$$

$$= -\varphi^T\gamma^0\gamma^i\psi = -\bar{\varphi}\gamma^i\psi \quad \text{(cont.)}$$

Thus for example,

$$(D_m \bar{\psi}) \gamma^n \psi = - \bar{\psi} \gamma^n (D_m \psi)$$

For Dirac spinors this fails because the spinors are complex. $(\bar{\psi} \gamma^n \varphi = -(\bar{\varphi} \gamma^n \psi)^*)$

RARITA-SCHWINGER FIELD
(in flat space)

ψ_m = real, vector-spinor

Massless case. Lagrangian:

$$L_{RS} = +\frac{i}{2} \epsilon^{lmnr} \bar{\psi}_l \gamma_5 \gamma_m \partial_n \psi_r$$

$$\epsilon_{0123} = +1 \qquad \gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 \qquad \bar{\psi}_l = \psi_l \gamma^0$$

The action is invariant under the gauge transformation,

$$\psi_r \to \psi_r + \partial_r \varphi$$

with φ = x dependant spinor

Get equations of motion by varying ψ:

$$R^{\ell} \overset{\text{def.}}{=} \epsilon^{\ell m n r} \gamma_5 \gamma_m \partial_n \psi_r = 0$$

This equation can be rewritten in a number of equivalent ways:

$$\gamma \cdot R = -\epsilon^{\ell m n r} \gamma_5 \gamma_\ell \gamma_m \partial_n \psi_r$$

$$= -\epsilon^{\ell m n r} \gamma_5 \Sigma_{\ell m} \psi_{nr}$$

$$\Sigma_{\ell m} = \tfrac{1}{4}[\gamma_\ell, \gamma_m]$$

$$\psi_{nr} = \partial_n \psi_r - \partial_r \psi_n =$$ Rarita-Schwinger field strength.

Two gamma matrix identities:

$$\gamma^\ell \gamma^m \gamma^n - \gamma^n \gamma^m \gamma^\ell = 2\epsilon^{\ell m n r} \gamma_5 \gamma_r \qquad (1)$$

$$\epsilon^{\ell m n r} \gamma_5 \Sigma_{\ell m} = -2 \Sigma^{nr} \qquad (2)$$

Using (1) we rewrite R^ℓ as:

$$R^\ell = -\tfrac{1}{2} \epsilon^{n \ell r m} \gamma_5 \gamma_m \psi_{nr}$$

$$= -\tfrac{1}{4}\left(\gamma^n \gamma^\ell \gamma^r - \gamma^r \gamma^\ell \gamma^n\right) \psi_{nr}$$

$$R^\ell = -\tfrac{1}{2}\gamma^n\gamma^\ell\gamma^r\psi_{nr}$$

$$= \left(\tfrac{1}{2}\gamma^\ell\gamma^n\gamma^r - \eta^{n\ell}\gamma^r\right)\psi_{nr}$$

$$= \gamma^\ell\Sigma^{nr}\psi_{nr} - \gamma^r\psi^\ell{}_r$$

Using (2) we have,

$$\Sigma^{nr}\psi_{nr} = -\tfrac{1}{2}\epsilon^{\ell mnr}\gamma_5\Sigma_{\ell m}\psi_{nr}$$

$$= \tfrac{1}{2}\gamma\cdot R$$

$$\underline{R^\ell = \tfrac{1}{2}\gamma^\ell(\gamma\cdot R) - \gamma_r\psi^{\ell r}}$$

$$\epsilon_{mtuv}\gamma_5 R^m = -\tfrac{1}{2}\epsilon_{mtuv}\epsilon^{mnrs}\gamma_n\psi_{rs}$$

$$= +\tfrac{1}{2}\Big(\delta^{nrs}_{tuv} + \delta^{nrs}_{uvt} + \delta^{nrs}_{vtu}$$

$$- \delta^{nrs}_{utv} - \delta^{nrs}_{vut} - \delta^{nrs}_{tvu}\Big)\gamma_n\psi_{rs}$$

$$= \gamma_t\psi_{uv} + \gamma_u\psi_{vt} + \gamma_v\psi_{tu}$$

In summary:

$$R^\ell = 0\ ,\quad \gamma_m\psi^{m\ell} = 0,\quad \oint_{tuv}\gamma_t\psi_{uv} = 0$$

SPECIAL GAUGE FOR LINEARIZED THEORY

We can show the existence of the gauge

$$\gamma \cdot \psi = 0$$

It will then follow from the equations of motion that also,

$$\partial \cdot \psi = 0$$

in this gauge.

$$\psi_s \longrightarrow \psi_s + \partial_s \varphi$$

$$\gamma \cdot \psi \longrightarrow \gamma \cdot \psi + (\gamma \cdot \partial)\varphi$$

Thus by solving the Dirac equation,

$$(\gamma \cdot \partial)\varphi = -\gamma \cdot \psi$$

for φ, we obtain the gauge $\gamma \cdot \psi = 0$.

From a particular form of the equations of motion,

$$\Sigma^{mn} \psi_{mn} = 0$$

$$(\gamma^m \gamma^n - \gamma^n \gamma^m)\partial_m \psi_n = 0$$

$$(2\gamma^m\gamma^n - 2\eta^{mn})\partial_m \psi_n = 0$$

$$(\gamma\cdot\partial)\gamma\cdot\psi - \partial\cdot\psi = 0$$

We see that $\gamma\cdot\psi = 0$ implies $\partial\cdot\psi = 0$.

If we use a different form of the equation of motion,

$$\gamma^m \psi_{mn} = \gamma^m(\partial_m\psi_n - \partial_n\psi_m) = 0$$

we see that in the gauge $\gamma\cdot\psi = 0$,

$$(\gamma\cdot\partial)\psi_n = 0 \qquad *$$

each component satisfies the massless Dirac equation.

Both gauge conditions,

$$\gamma\cdot\psi = \partial\cdot\psi = 0$$

are required to show the field describes a particle of helicity $\pm 3/2$.

---

* This equation (+ gauge condition) also implies $\partial\cdot\psi = 0$, check!

RARITA-SCHWINGER FIELD IN CURVED SPACE

Two changes are necessary:

In the lagrangian replace,

$$\partial_n \psi_r \longrightarrow \tilde{D}_n \psi_r$$

Antisymmetry in (n,r) insures the action is still generally covariant. Invariance under local lorentz transformations (which rotate each spinor component of the vector ψ_r) is provided by the minimal covariant derivative,

$$\tilde{D}_n \psi_r \overset{\text{def.}}{\equiv} \partial_n \psi_r - \tfrac{1}{2}\omega_{nab}\Sigma^{ab}\psi_r$$

The complete covariant derivative could also have been used to write an invariant action, however this describes a different theory in the same sense that covariant derivatives are <u>not</u> used in the Maxwell action to carry E.M. into curved space.

The other thing to check is that the lagrangian is a density of weight 1. This happens automatically if we replace the flat space indices with world-type indices since then the Levi-civita tensor becomes a tensor density of weight 1:

$$\epsilon^{lmnr} = e\, \epsilon^{abcd}\, e_a{}^l\, e_b{}^m\, e_c{}^n\, e_d{}^r$$

(To check this note that the RHS is completely antisymmetric in (l,m,n,r). Choosing a particular ordering ($l=0$, $m=1$, $n=2$, $r=3$) we see that the RHS just gives $e \cdot \det(e_a{}^l)$ $= e \cdot e^{-1} = 1$. The LHS is a density because the RHS obviously is.)

Thus the action can simply be written as:

$$I = \int L_{RS}\, d^4x$$

$$L_{RS} = \frac{i}{2}\, \epsilon^{lmnr}\, \bar{\psi}_l\, \gamma_5\, \gamma_m\, \tilde{D}_n\, \psi_r$$

The equations of motion are obtained by varying ψ_l. The manipulations

involved in the derivation will require that we understand how the product rule for $\tilde{D}$ works on spinors. A simple example will make this clear:

$\bar{\psi}\gamma^a\varphi$ transforms like v^a

so we expect,

$$\tilde{D}_m(\bar{\psi}\gamma^a\varphi) = \partial_m(\bar{\psi}\gamma^a\varphi) + (\bar{\psi}\gamma^b\varphi)\omega_{mb}{}^a$$

If we apply the product rule only to the <u>spinors</u>,

$$\tilde{D}_m(\bar{\psi}\gamma^a\varphi) = (\overline{\tilde{D}_m\psi})\gamma^a\varphi + \bar{\psi}\gamma^a(\tilde{D}_m\varphi)$$

$$= \left(\partial_m\bar{\psi} - \tfrac{1}{2}\omega_{mcb}\overline{\Sigma^{cb}\psi}\right)\gamma^a\varphi$$

$$+ \bar{\psi}\gamma^a\left(\partial_m\varphi - \tfrac{1}{2}\omega_{mcb}\Sigma^{cb}\varphi\right)$$

$$= \partial_m(\bar{\psi}\gamma^a\varphi) - \tfrac{1}{2}\omega_{mcb}\,\bar{\psi}(-\Sigma^{cb}\gamma^a + \gamma^a\Sigma^{cb})\varphi$$

$$= \partial_m(\bar{\psi}\gamma^a\varphi) - \tfrac{1}{2}\bar{\psi}(\omega_m{}^a{}_b\gamma^b - \omega_{mc}{}^a\gamma^c)\varphi$$

$$= \partial_m(\bar{\psi}\gamma^a\varphi) + \omega_{mc}{}^a\bar{\psi}\gamma^c\varphi$$

We get the correct result.

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$$\delta L_{RS} = \frac{i}{2}\epsilon^{lmnr}\left\{\overline{\delta\psi_l}\,\gamma_5\gamma_m\tilde{D}_n\psi_r + \overline{\psi_l}\,\gamma_5\gamma_m\tilde{D}_n(\delta\psi_r)\right\}$$

The second term can be transformed into the first term plus a term involving the torsion and a divergence:

$$\partial_n\left\{\epsilon^{lmnr}\,\overline{\delta\psi_l}\,\gamma_5\gamma_m\psi_r\right\} = \underbrace{\epsilon^{lmnr}}_{\text{constant}}\partial_n\{\quad\}$$

$$= \epsilon^{lmnr}\tilde{D}_n\{\quad\} = \epsilon^{lmnr}\overline{(\tilde{D}_n\delta\psi_l)}\gamma_5\gamma_m\psi_r$$

$$+ \quad '' \quad \overline{\delta\psi_l}\,\gamma_5\gamma_a(\tilde{D}_n e_m{}^a)\psi_r$$

$$+ \quad '' \quad \overline{\delta\psi_l}\,\gamma_5\gamma_m(\tilde{D}_n\psi_r)$$

$$= \epsilon^{lmnr}\left\{-\overline{\psi_l}\,\gamma_5\gamma_m\tilde{D}_n\delta\psi_r + \frac{1}{2}(\overline{\delta\psi_l}\,\gamma_5\gamma_a\psi_r)T_{nm}{}^a + \overline{\delta\psi_l}\,\gamma_5\gamma_m(\tilde{D}_n\psi_r)\right\}$$

$$\delta L_{RS} = \frac{i}{2}\epsilon^{lmnr}\left\{2\,\overline{\delta\psi_l}\,\gamma_5\gamma_m(\tilde{D}_n\psi_r) + \frac{1}{2}\overline{\delta\psi_l}\,\gamma_5\gamma_a\psi_r\,T_{nm}{}^a\right\}$$

$$+ \text{(div.)}$$

$$R^{\ell} \stackrel{\text{def.}}{=} \epsilon^{\ell mnr}\left\{ \gamma_5 \gamma_m \tilde{D}_n \psi_r + \frac{1}{4} T_{nm}{}^{a} \gamma_5 \gamma_a \psi_r \right\}$$

$R^{\ell} = 0$ are the equations of motion

When gravity is coupled to the Rarita-Schwinger field we have the lagrangian:

$$L = -\frac{1}{2} e R + \frac{i}{2} \epsilon^{\ell mnr} \bar{\psi}_{\ell} \gamma_5 \gamma_m \tilde{D}_n \psi_r$$

The only place the vierbein appears in the R.S. piece is in the combination $\gamma_m = \gamma_a e_m{}^a$. Thus, refering to page 20 of the supernotes, the equation of motion for the vierbein is:

$$e\left(R_a{}^m - \frac{1}{2} e_a{}^m R\right) = -\frac{i}{2} \epsilon^{\ell mnr} \bar{\psi}_{\ell} \gamma_5 \gamma_a \tilde{D}_n \psi_r$$

$$e\, G_a{}^m = \quad ''$$

Note: $\delta(e_m{}^a) = -\delta(e_b{}^n)\, e_n{}^a e_m{}^b$

Next we vary the connection $\omega_n{}^{ab}$:

$$\delta(R.S.) = -\frac{i}{4}\epsilon^{lmnr}\left(\bar{\psi}_l \gamma_5 \gamma_m \Sigma_{ab} \psi_r\right)\delta\omega_n{}^{ab}$$

$$\gamma_c \Sigma_{ab} = \underbrace{\frac{1}{2}[\gamma_c, \Sigma_{ab}]}_{\propto \gamma_d,\ \bar{\psi}_l \gamma_5 \gamma_d \psi_r = \text{symmetric}} + \frac{1}{2}\{\gamma_c, \Sigma_{ab}\}$$

so we keep only the anticommutator.

$$\{\gamma_c, \Sigma_{ab}\} = \epsilon_{cabd}\gamma_5\gamma^d$$

$$\delta(R.S.) = \frac{i}{4}\epsilon^{lmnr}\bar{\psi}_l\left(e_m{}^c \tfrac{1}{2}\epsilon_{cabd}\gamma^d\right)\psi_r\,\delta\omega_n{}^{ab}$$

$$= -\frac{i}{8}\left(\epsilon^{cefg}\epsilon_{cabd}\right)\left(\bar{\psi}_e\gamma^d\psi_g\right)\delta\omega_f{}^{ab}e$$

$$= \frac{i}{4}\left\{\left(\bar{\psi}_a\gamma^d\psi_d\right)\delta^f{}_b + \bar{\psi}_b\gamma^f\psi_a + \left(\bar{\psi}_d\gamma^d\psi_b\right)\delta^f{}_a\right\} \cdot \delta\omega_f{}^{ab}e$$

The variation of the Einstein part (p. 20) gives:

$$\delta\left(-\tfrac{1}{2}eR\right) = -\frac{1}{2}\left\{T_{ab}{}^f - T_{ad}{}^d\delta^f{}_b + T_{bd}{}^d\delta^f{}_a\right\} \cdot \delta\omega_f{}^{ab}e$$

Taking the trace on b, f in the resulting

equation of motion gives :

$$\frac{i}{4}(4-1-1)\,\bar{\psi}_a \gamma^d \psi_d \quad +$$

$$-\frac{1}{2}(1-4+1)\,T_{ad}{}^{d} \quad = 0$$

or $\quad T_{ad}{}^{d} = -\frac{i}{2}\,\bar{\psi}_a \gamma^d \psi_d$

Substituting into the untraced equations gives:

$$T_{ab}{}^{c} = -\frac{i}{2}\,\bar{\psi}_a \gamma^c \psi_b$$

As in the spin 1/2 case (p. 25) we can now show that the term in the equation of motion for ψ_r involving the torsion identically vanishes. This term is proportional to,

$$(\bar{\psi}_a \gamma^c \psi_b)\,\gamma_c\,\psi_d$$

where antisymmetry in (a, b, d) is understood. We use the Fierz

rearrangement formula:

$$(\bar{\alpha}\chi)\beta = -\frac{1}{4}\sum_A (\bar{\alpha}\gamma_A\beta)\gamma^A\chi$$

where $\alpha = -\gamma^c\psi_a$, $\beta = \gamma_c\psi_d$, $\chi = \psi_b$

$$(\bar{\psi}_a\gamma^c\psi_b)\gamma_c\psi_d = -\frac{1}{4}\Big\{\cancel{(\bar{\psi}_a\gamma^c\gamma_c\psi_d)}\psi_b$$

$$-\cancel{(\bar{\psi}_a\gamma^c\gamma_5\gamma_c\psi_d)}\gamma_5\psi_b + \cancel{(\bar{\psi}_a\gamma^c\gamma_5\gamma_e\gamma_c\psi_d)}\gamma_5\gamma^e\psi_b$$

$$+(\bar{\psi}_a\gamma^c\gamma_e\gamma_c\psi_d)\gamma^e\psi_b - \underset{(e<f)}{\cancel{(\bar{\psi}_a\gamma^c\gamma_e\gamma_f\gamma_c\psi_d)}}\gamma^e\gamma^f\psi_b\Big\}$$

First three terms vanish by antisymmetry in (a,d); $(\gamma^c\gamma_5\gamma_e\gamma_c = 2\gamma_5\gamma_f)$

The last term vanishes identically,

$$\gamma^c\gamma_e\gamma_f\gamma_c = 0 \qquad e \neq f$$

That gives,

$$(\bar{\psi}_a\gamma^c\psi_b)\gamma_c\psi_d = \frac{1}{2}(\bar{\psi}_a\gamma^e\psi_d)\gamma_e\psi_b$$

or $\quad (\bar{\psi}_a\gamma^c\psi_b)\gamma_c\psi_d = 0$

WHAT IS MEANT BY CONSISTENCY?

EXAMPLE: Massless complex spin one field.

$$W_{mn} = \partial_m W_n - \partial_n W_m$$

$$\partial^m W_{mn} = 0$$

Couple to the an external electromagnetic potential A_m by the substitution

$$\partial_m \longrightarrow \partial_m + ie A_m \overset{Def}{=} \mathcal{D}_m$$

Now

$$W_{mn} = \mathcal{D}_m W_n - \mathcal{D}_n W_m$$

$$\mathcal{D}^m W_{mn} = 0$$

Apply $\mathcal{D}^n$ to the last eq. Since

$$[\mathcal{D}_m, \mathcal{D}_n] = ie(\partial_m A_n - \partial_n A_m) = ie F_{mn}$$

one has

$$e F^{mn} W_{mn} = 0$$

For $e \neq 0$ this gives a constraint which is absent in the free field case. There may be solutions, but the solutions of the coupled system are "fewer" than those of the free system. Consequences for quantization. This can be remedied by adding a "non minimal" coupling as in Yang-Mills theory. Check consistency.

CONSISTENCY OF THE COUPLED EINSTEIN / RARITA-SCHWINGER SYSTEM.

The consistency problem arises when we consider $\tilde{D}_\ell R^\ell$ and find it is proportional to a particular combination of the Einstein tensor and the torsion. This appears to impose an algebraic constraint on the other variables of the theory (vierbein, connection). However, we will see that this particular combination vanishes identically as a consequence of the other equations of motion.

$$R^\ell = \epsilon^{\ell mnr} \{ \gamma_5 \gamma_m \tilde{D}_n \psi_r + \tfrac{1}{4} T_{nm}{}^a \gamma_5 \gamma_a \psi_r \}$$

(as we've seen, the second term actually vanishes; however we don't want to use the equations of motion yet, so we'll keep it.)

$$\tilde{D}_\ell R^\ell = \epsilon^{\ell mnr} \gamma_5 \Big\{ (\tilde{D}_\ell e_m{}^a) \gamma_a \tilde{D}_n \psi_r \; + \; \gamma_m (\tilde{D}_\ell \tilde{D}_n \psi_r) + \tfrac{1}{4} (\tilde{D}_\ell T_{nm}{}^a) \gamma_a \psi_r \; + \; \tfrac{1}{4} T_{nm}{}^a \gamma_a (\tilde{D}_\ell \psi_r) \Big\}$$

The first and last terms combine, the second splits into two terms:

$$\underbrace{\gamma_m (\tilde{D}_\ell \tilde{D}_n \psi_r)}_{\text{antisym.}} = \tfrac{1}{2} \gamma_m [\tilde{D}_\ell, \tilde{D}_n] \psi_r$$

$$= \tfrac{1}{2} \gamma_m \left(-\tfrac{1}{2}\right) R_{\ell n}{}^{ab} \Sigma_{ab} \psi_r$$

$$\gamma_c \Sigma_{ab} = \tfrac{1}{2} [\gamma_c, \Sigma_{ab}] + \tfrac{1}{2} \{\gamma_c, \Sigma_{ab}\}$$

$$= \tfrac{1}{2} (\gamma_b \eta_{ac} - \gamma_a \eta_{bc})$$

$$+ \tfrac{1}{2} \epsilon_{cabd} \gamma_5 \gamma^d$$

$$= -\tfrac{1}{8} \Big\{ \underbrace{e_m{}^c R_{\ell n}{}^{ab} (\gamma_b \eta_{ac} - \gamma_a \eta_{bc}) \psi_r}_{2 R_{\ell n m}{}^b \gamma_b \psi_r} \Big\}$$

$$+ \; (\epsilon \text{ term})$$

The ϵ term combines with the other ϵ to make an Einstein tensor out of the curvature:

$$\epsilon^{lmnr}\gamma_5\left(-\tfrac{1}{8}\right)\epsilon_{cabd}\, e_m{}^c R_{ln}{}^{ab}\gamma_5\gamma^d\psi_r$$

$$= -\tfrac{1}{8}e\,\epsilon^{cefg}\epsilon_{cabd}R_{ef}{}^{ab}\gamma^d\psi_g$$

$$= \tfrac{1}{4}e\left\{R_{ab}{}^{ab}\gamma^d\psi_d + R_{bd}{}^{ab}\gamma^d\psi_a + R_{da}{}^{ab}\gamma^d\psi_b\right\}$$

$$= -\tfrac{e}{2}\left\{R_a{}^m - \tfrac{1}{2}e_a{}^m R\right\}\gamma^a\psi_m$$

$$= -\tfrac{e}{2}G_a{}^m\gamma^a\psi_m$$

$$\tilde{D}_l R^l = -\tfrac{e}{2}G_a{}^m\gamma^a\psi_m + \tfrac{1}{4}\epsilon^{lmnr}\gamma_5 T_{lm}{}^a\gamma_a\tilde{D}_n\psi_r$$

$$+\tfrac{1}{4}\epsilon^{lmnr}\gamma_5\left\{\underbrace{(\tilde{D}_l T_{nm}{}^a)\gamma_a\psi_r - R_{lnm}{}^a\gamma_a\psi_r}\right\}$$

Fortunately, this term vanisles

due to a Bianchi identity (if it didn't, we couldn't substitute for the un-contracted curvature tensor)

Recall (p.3):

$$\oint_{rmn} \left(D_r T_{mn}{}^a + T_{rm}{}^s T_{sn}{}^a - R_{rmn}{}^a \right) = 0$$

(note: full covariant derivative; a = tangent space index!)

$$\oint_{rmn} D_r T_{mn}{}^a = \oint_{rmn} \left(\tilde{D}_r T_{mn}{}^a - \Gamma_{rm}{}^t T_{tn}{}^a - \Gamma_{rn}{}^t T_{mt}{}^a \right)$$

$$= \oint_{rmn} \left(\tilde{D}_r T_{mn}{}^a - T_{rm}{}^t T_{tn}{}^a \right)$$

Thus: $$\oint_{rmn} \left(\tilde{D}_r T_{mn}{}^a - R_{rmn}{}^a \right) = 0$$

$$\tilde{D}_\ell R^\ell = -\frac{e}{2} G_a{}^m \gamma^a \psi_m + \frac{1}{4} \epsilon^{\ell mnr} \gamma_5 T_{\ell m}{}^a \gamma_a \tilde{D}_n \psi_r$$

We now substitute the RHS's of the

other equations of motion:

$$\tilde{D}_\ell R^\ell = \frac{i}{4}\,\epsilon^{\ell m n r}(\bar{\psi}_\ell \gamma_5 \gamma_a \tilde{D}_n \psi_r)\gamma^a \psi_m \quad (p.40)$$

$$-\frac{i}{8}\,\epsilon^{\ell m n r}(\bar{\psi}_\ell \gamma^a \psi_m)\gamma_5 \gamma_a \tilde{D}_n \psi_r \quad (p.42)$$

Fierz rearrange the first term:

$$(\bar{\psi}_\ell \gamma_5 \gamma_a \tilde{D}_n \psi_r)\gamma^a \psi_m = \quad (\text{by antisym. in } (\ell, m))$$

$$-\frac{1}{4}(\bar{\psi}_\ell \gamma_5 \gamma_a \gamma_5 \gamma_b \gamma^a \psi_m)\gamma_5 \gamma^b \tilde{D}_n \psi_r$$

$$= \frac{1}{2}(\bar{\psi}_\ell \gamma_b \psi_m)\gamma_5 \gamma^b \tilde{D}_n \psi_r$$

This precisely cancels the second term.

Thus, $\tilde{D}_\ell R^\ell = 0$

Physics 250 Bruno Zumino

Sequel to page 49:

The invariance of the flat space Rarita-Schwinger action under gauge transformations becomes the supergravity invariance of the coupled R.S. / Einstein system. The existence of this invariance is related to the "consistency" of the coupled equations.

$$I = \int d^4x \left\{ \frac{-1}{2K^2} eR + \frac{i}{2} \epsilon^{lmnr} \bar{\psi}_l \gamma_5 \gamma_m \tilde{D}_n \psi_r \right\}$$

$$\left.\begin{aligned} \delta e_m{}^a &= -iK(\bar{\alpha}\gamma^a \psi_m) \\ \delta \psi_l &= \frac{2}{K} \tilde{D}_l \alpha \\ \delta \omega_m{}^{ab} &= B_m{}^{ab} - \tfrac{1}{2} e_m{}^b B_c{}^{ac} + \tfrac{1}{2} e_m{}^a B_c{}^{bc} \\ B_a{}^{lm} &= \frac{-iK}{e} \epsilon^{lmnr} (\bar{\alpha} \gamma_5 \gamma_a \tilde{D}_n \psi_r) \end{aligned}\right\} \text{supergravity transfs.}$$

(K = gravitational coupling ~ length)

$$\delta I = \int d^4x \left\{ \frac{\delta I}{\delta e_m{}^a} \delta e_m{}^a + \frac{\delta I}{\delta \omega_m{}^{ab}} \delta \omega_m{}^{ab} + \frac{\delta I}{\delta \psi_l} \delta \psi_l \right\}$$

from page 40, the first term looks like,

$$\sim (B + \psi^2)(\alpha \psi)$$

from p. 41, the second term is:

$$\sim (T + \psi^2)(\alpha\psi)$$

the third term, p. 39–40:

$$\sim (\overline{\tilde{D}_\ell \alpha})\, R^\ell = -\bar{\alpha}(\tilde{D}_\ell R^\ell) \sim \alpha(G\psi + T\psi)$$

by p. 48 ("consistency"). Thus,

$$\delta I \sim (G + \psi^2)(\alpha\psi) + (T + \psi^2)(\alpha\psi) + \alpha(G\psi + T\psi)$$

$$= 0$$

Details of the calculation:

$$\frac{\delta I}{\delta e_m{}^a}\,\delta e_m{}^a = -\frac{i}{K} e\, G_a{}^m (\bar{\alpha}\gamma^a \psi_m)$$

$$+ \frac{K}{2}\, \epsilon^{\ell m n r} (\bar{\alpha}\gamma^a \psi_m)(\bar{\psi}_\ell \gamma_5 \gamma_a \tilde{D}_n \psi_r)$$

The second term simplifies due to the identity,

$$(T_{ab}{}^c - T_{ad}{}^d \delta^c_b + T_{bd}{}^d \delta^c_a) \cdot$$

$$\cdot \left(B_c{}^{ab} - \tfrac{1}{2} B_d{}^{ad} \delta_c{}^b + \tfrac{1}{2} B_d{}^{bd} \delta^a_c\right)$$

$$= T_{ab}{}^c B_c{}^{ab}$$

provided T and B are antisymmetric in lower and upper two indices.

$$\frac{\delta I}{\delta \omega_m{}^{ab}}\,\delta\omega_m{}^{ab} = \frac{-e}{2K^2}\underbrace{T_{ab}{}^m B_m{}^{ab}}_{T_{\ell m}{}^a B_a{}^{\ell m}} - \frac{ie}{4}(\bar\psi_{\underset{\ell}{a}}\gamma^{\underset{a}{m}}\psi_{\underset{m}{b}})B_{\underset{a}{m}}{}^{\underset{\ell m}{ab}}$$

$$= \frac{i}{2K}T_{\ell m}{}^a\,\epsilon^{\ell mnr}(\bar\alpha\gamma_5\gamma_a\tilde D_n\psi_r)$$

$$-\frac{K}{4}\epsilon^{\ell mnr}(\bar\psi_\ell\gamma^a\psi_m)(\bar\alpha\gamma_5\gamma_a\tilde D_n\psi_r)$$

Finally, $\int d^4x\,\frac{\delta I}{\delta\psi_\ell}\delta\psi_\ell = \int d^4x\,\frac{2}{K}\overline{\tilde D_\ell\alpha}\;iR^\ell$

$$= \int d^4x\left(-\frac{2i}{K}\right)(\bar\alpha\,\tilde D_\ell R^\ell)$$

$$= \int d^4x\left\{\frac{ie}{K}G_a{}^m(\bar\alpha\gamma^a\psi_m)\right.$$

$$\left.-\frac{i}{2K}T_{\ell m}{}^a\,\epsilon^{\ell mnr}(\bar\alpha\gamma_5\gamma_a\tilde D_n\psi_r)\right\}$$

The four terms linear in ψ cancel. The remaining cubic terms are:

$$\frac{K}{2}\epsilon^{\ell mnr}\left\{(\bar\alpha\gamma^a\psi_m)(\bar\psi_\ell\gamma_5\gamma_a\tilde D_n\psi_r)\right.$$

$$\left.-\frac{1}{2}(\bar\psi_\ell\gamma^a\psi_m)(\bar\alpha\gamma_5\gamma_a\tilde D_n\psi_r)\right\}$$

Using the formula,

$$(\bar\phi\chi)(\bar\psi\eta) = -\frac{1}{4}\sum_A(\bar\phi\gamma_A\eta)(\bar\psi\gamma^A\chi)$$

with $\bar\phi = \bar\alpha\gamma^a$, $\chi = \psi_m$, $\bar\psi = \bar\psi_\ell$, $\eta = \gamma_5\gamma_a\tilde D_n\psi_r$

the first term is easily shown to be the negative of the second.

MODIFIED SUPERGRAVITY

To the supergravity lagrangian we can add,

$$ce + \frac{i}{2} M \epsilon^{lmnr} \bar{\psi}_l \gamma_5 \Sigma_{mn} \psi_r$$

This theory is still locally supersymmetric provided

$$e K^2 = 3 M^2 \qquad (*)$$

and the supersymmetry transformations are slightly modified with the replacement

$$\tilde{D}_n \longrightarrow \hat{D}_n = \tilde{D}_n + \frac{1}{2} M \gamma_n$$

for the spinor covariant derivatives. In fact, the total R.S. lagrangian may be written compactly as,

$$L_{RS} = \frac{i}{2} \epsilon^{lmnr} \bar{\psi}_l \gamma_5 \gamma_m \hat{D}_n \psi_r$$

To check $*$ above, only consider the variation of $L_{RS} \propto M^2$:

$$\delta(ce) = c\, e_a{}^m (-iK\bar{\alpha}\gamma^a \psi_m)$$
$$= -icK\,(\bar{\alpha}\gamma^a \psi_a)$$

$$\delta\left(\tfrac{i}{2}M\epsilon^{lmnr}\bar{\psi}_l \gamma_5 \Sigma_{mn}\psi_r\right) = iM\epsilon^{lmnr}(\delta\psi_l \gamma_5 \Sigma_{mn}\psi_r)$$
$$+ (\ )(\delta e_m{}^a) \leftarrow \mathcal{O}(M)$$

$$= \frac{2iM}{K}\epsilon^{lmnr}\left(\overline{\hat{D}_l \alpha}\,\gamma_5 \Sigma_{mn}\psi_r\right) + \cdots$$

$$= \frac{2iM}{K}\epsilon^{lmnr}\left(\tfrac{1}{2}M\overline{\gamma_l \alpha}\,\gamma_5 \Sigma_{mn}\psi_r\right) + \cdots$$

$$= \frac{-iM^2}{K}\epsilon^{lmnr}\Big(\bar{\alpha}\,\underbrace{\gamma_l \Sigma_{mn}}_{\frac{1}{2}\epsilon_{lmnt}\gamma_5\gamma^t}\gamma_5\psi_r\Big) + \cdots$$

$$= \frac{-iM^2}{2K}\underbrace{(\epsilon^{lmnr}\epsilon_{lmnt})}_{-6\delta^r{}_t}(\bar{\alpha}\gamma^t\psi_r) = \frac{3iM^2}{K}(\bar{\alpha}\gamma^a\psi_a)$$
$$+ \cdots$$

these cancel if $c = \dfrac{3M^2}{K}$.

When this relation holds so that local supersymmetry is still exact, then the Rarita-Schwinger field is still <u>massless</u> in spite of the apparent mass term. This is a consequence

of quantizing the R.S. field in a background spacetime of nonzero cosmological constant (De Sitter Space) where this "mass" term no longer has the same interpretation.

VOLKOV–AKULOV MODEL

(Rigid supersymmetry)

This is a generic model of spontaneously broken SUSY in the sense of the non-linear sigma model.

$\lambda =$ Majorana spinor ("Goldstone" spinor, "Goldstino")

$\alpha =$ infinitesimal constant parameter

$f =$ constant with dim. mass^{-2}

$$\delta\lambda = \frac{1}{f}\alpha + if(\bar{\alpha}\gamma^a\lambda)\partial_a\lambda$$

$$\begin{aligned}[\delta_2,\delta_1]\lambda = {} & 2i(\bar{\alpha}_1\gamma^a\alpha_2)\partial_a\lambda - \\ & - f^2\Big\{(\bar{\alpha}_1\gamma^a\partial_b\lambda)(\bar{\alpha}_2\gamma^b\lambda)\partial_a\lambda + \\ & + (\bar{\alpha}_1\gamma^a\lambda)(\bar{\alpha}_2\gamma^b\partial_a\lambda)\partial_b\lambda + \\ & + (\bar{\alpha}_1\gamma^a\lambda)(\bar{\alpha}_2\gamma^b\lambda)\partial_a\partial_b\lambda \\ & - (1 \leftrightarrow 2)\Big\}\end{aligned}$$

The last { } vanishes identically so the commutator of two infinitesimal SUSY transfs. is just a translation.

The appropriately normalized lagrangian, invariant under this transformation, is:

$$\mathcal{L}_{V.A.} = \frac{-1}{2f^2} \det\left\{ \delta_a{}^b + if^2(\bar{\lambda}\gamma^b\partial_a\lambda)\right\}$$

$$= \underbrace{-\frac{1}{2f^2}}_{\text{non zero VEV}} - \frac{i}{2}\bar{\lambda}\gamma\cdot\partial\lambda + \cdots$$

Check of invariance:

$$M_m{}^a \overset{\text{Def.}}{=} \delta_m{}^a + if^2(\bar{\lambda}\gamma^a\partial_m\lambda)$$

$$\delta M_m{}^a = if^2\Big[\tfrac{1}{f}\overset{③}{(\bar{\alpha}\gamma^a\partial_m\lambda)} + if\overset{①}{(\bar{\alpha}\gamma^b\lambda)}(\partial_b\bar{\lambda}\gamma^a\partial_m\lambda)$$

$$+ if\overset{②}{(\bar{\alpha}\gamma^b\lambda)}(\bar{\lambda}\gamma^a\partial_m\partial_b\lambda) + if\overset{④}{(\bar{\alpha}\gamma^b\partial_m\lambda)}(\bar{\lambda}\gamma^a\partial_b\lambda)\Big]$$

Now, $if(\bar{\alpha}\gamma^b\lambda)\partial_b M_m{}^a = -f^3(\bar{\alpha}\gamma^b\lambda)\Big[\overset{①}{(\partial_b\bar{\lambda}\gamma^a\partial_m\lambda)} + \overset{②}{(\bar{\lambda}\gamma^a\partial_b\partial_m\lambda)}\Big]$

$$if\,\partial_m(\bar{\alpha}\gamma^b\lambda)M_b{}^a = if(\bar{\alpha}\gamma^b\partial_m\lambda)\Big[\underset{③}{\delta_b{}^a} + \underset{④}{if^2(\bar{\lambda}\gamma^a\partial_b\lambda)}\Big]$$

The sum of these matches the preceeding line. Thus:

$$\delta M_m{}^a = \xi^\ell \partial_\ell M_m{}^a + (\partial_m \xi^\ell) M_\ell{}^a$$

where: $\xi^\ell = i f (\bar{\alpha} \gamma^\ell \lambda)$

This is the transformation law of the vierbein under general coord. transf's. Therefore,

$$M \overset{\text{Def.}}{=} \det \{ M_m{}^a \}$$

transforms as:

$$\delta M = \partial_\ell \{ \xi^\ell M \}$$

SUPER GRAVITY (SUGRA) COUPLED TO *SUPERSYMMETRIC MATTER

(* spontaneously broken)

The complete coupling of the Volkov–Akulov model to SUGRA has been done but it is not easy. However, the first few terms in powers of K and f are easy to find:

$$L = \underbrace{-\frac{1}{2K^2} eR + \frac{i}{2} \epsilon^{\ell m n r} \bar{\psi}_\ell \gamma_5 \gamma_m \hat{D}_n \psi_r + ce}_{\text{modified SUGRA}}$$

$$\underset{①}{-\frac{1}{2f^2} e} \underset{②}{- \frac{i}{2} \bar{\lambda} \gamma \cdot \partial \lambda} \underset{③}{+ \frac{iK}{2f} \bar{\lambda} \gamma \cdot \psi} \underset{④}{+ i M \bar{\lambda} \lambda} + \cdots$$

"$+\cdots$" means terms higher than quadratic in the fields. Terms ① and ② are the covariantized first two terms of L.V.A.; term ③ is needed since $\alpha(x)$ is now a local parameter; ④ comes from the modification of SUGRA.

$$\delta e_m{}^a = -iK\bar{\alpha}\gamma^a \psi_m$$

$$\delta \omega_m{}^{ab} = \cdots$$

$$\delta \psi_m = \frac{2}{K}\left(\partial_m \alpha + \frac{1}{2} M \gamma_m \alpha\right) + \cdots$$

$$\delta \lambda = \frac{1}{f}\alpha(x) + \cdots$$

The terms included above are enough to check those written in the lagrangian.

SUSY HIGGS EFFECT

Exact cancellation of the two cosmological constants will give the gravitino (Rarita-Schwinger field) a true mass ($=M$). Since experimentally the cosmological constant is small, this gives a relationship between the scale of spontaneous SUSY breaking and the gravitino mass.

$$c = \frac{-1}{2f^2}\left(\begin{matrix}\text{spont.}\\ \text{Susy breaking}\end{matrix}\right) \qquad c = \frac{+3M^2}{K^2}\left(\begin{matrix}\text{modified}\\ \text{SUGRA}\end{matrix}\right)$$

cancellation $\Longrightarrow$ $6M^2 = \left(\frac{K}{f}\right)^2$

$$\frac{1}{f} = \sqrt{6}\,\frac{M}{K} = (\sqrt{6} \times 10^{19})\, M \quad (\mathrm{GeV})^2$$

For $M = 1$ keV $\quad \frac{1}{f} \sim 10^{13}\,(\mathrm{GeV})^2$

For $M = 10^4$ GeV $\quad \frac{1}{f} \sim 10^{23}\,(\mathrm{GeV})^2$

Cosmological arguments indicate that the gravitino mass should be either $\lesssim 1$ keV (Pagels, Primack) or $\gtrsim 10^4$ GeV (Weinberg).

Super Space

Four kinds of indices:

	bosonic	fermionic
tangent space	$a, b, c, \ldots$	$\alpha, \beta, \gamma, \ldots$
world	$m, n, \ell, \ldots$	$\mu, \nu, \lambda, \ldots$

Coordinates: $Z^M = (x^m, \theta^\mu)$

commuting $\quad$ anticommuting

Supervierbein: $E_M{}^A(Z) = \left(\begin{array}{c|c} E_m{}^a & E_m{}^\alpha \\ \hline E_\mu{}^a & E_\mu{}^\alpha \end{array}\right)$ 8×8 matrix

diagonal blocks: commuting

off-diagonal blocks: anticommuting

from: Recent Developments in Gravitation
Eds. M. Levy, S. Deser

B. Superdeterminants.

Let M be a matrix of the form

$$M = \begin{pmatrix} A & \Gamma \\ \Delta & B \end{pmatrix} \tag{B.1}$$

where the submatrices A and B have bosonic matrix elements while Γ and Δ have fermionic elements (respectively even and odd elements of a Grassmann algebra). Matrices like M can be combined linearly and multiplied with each other and the generic matrix of this kind has an inverse. The supervielbein matrix is of this type. We define the trace by

$$\mathrm{Tr}\, M = \mathrm{Tr}\, A - \mathrm{Tr}\, B \tag{B.2}$$

The reason for the minus sign in this definition is that we wish the basic property of the trace

$$\mathrm{Tr}\, M_1 M_2 = \mathrm{Tr}\, M_2 M_1 \tag{B.3}$$

to hold. Indeed we have

$$\begin{aligned} \mathrm{Tr}\, M_1 M_2 &= \mathrm{Tr}\left(A_1 A_2 + \Gamma_1 \Delta_2\right) - \mathrm{Tr}\left(\Delta_1 \Gamma_2 + B_1 B_2\right) \\ &= \mathrm{Tr}\left(A_2 A_1 - \Delta_2 \Gamma_1\right) - \mathrm{Tr}\left(-\Gamma_2 \Delta_1 + B_2 B_1\right) \\ &= \mathrm{Tr}\, M_2 M_1 \end{aligned} \tag{B.4}$$

The determinant can be defined from

$$\det M = \exp \mathrm{Tr}\, \ln M \tag{B.5}$$

Because of (B.3), it satisfies the product property

$$\det\left(M_1 M_2\right) = \left(\det M_1\right)\left(\det M_2\right) \tag{B.6}$$

Indeed, writing

$$\ln M = N\,, \tag{B.7}$$

one has

$$\ln(M_1 M_2) = N_1 + N_2 + \frac{1}{2}[N_1, N_2] + \cdots \tag{B.8}$$

where the dots denote multiple commutators. Taking the trace, all commutator terms vanish by (B.3), and one has

$$\mathrm{Tr}\ln(M_1 M_2) = \mathrm{Tr}\ln M_1 + \mathrm{Tr}\ln M_2 \tag{B.9}$$

which establishes (B.6). It follows from (B.5) that, if

$$M = 1 + X \tag{B.10}$$

where 1 is the unit matrix and X is infinitesimal then

$$\det M = 1 + \mathrm{Tr}\, X \tag{B.11}$$

More generally, one finds in the standard way, combining (B.6) and (B.11), that

$$\delta \det M = (\det M)\, \mathrm{Tr}\, M^{-1}\delta M \tag{B.12}$$

For any infinitesimal variation δM. An explicit form for the determinant can be obtained by writing M in the standard form

$$M = \begin{pmatrix} C & \Sigma \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \phi & D \end{pmatrix} \tag{B.13}$$

comparing with (B.1) one finds

$$\begin{aligned} C &= A - \Gamma B^{-1}\Delta, & \Sigma &= \Gamma . B^{-1} \\ \phi &= \Delta, & D &= B \end{aligned} \tag{B.14}$$

on the other hand (B.13) gives, using (B.2), (B.5) and (B.6),

$$\det M = \det C / \det D \tag{B.15}$$

Therefore

$$\det M = \frac{\det(A - \Gamma B^{-1}\Delta)}{\det B} \tag{B.16}$$

An equivalent form is

$$\det M = \frac{\det A}{\det(B - \Delta A^{-1}\Gamma)} \tag{B.17}$$

In (B.16) and (B.17) the determinants occurring in the right hand side are ordinary determinants of matrices with bosonic elements.

It is now easy to see that the determinant of the supervielbein matrix

$$E = \det E_M{}^A \tag{B.10}$$

transforms like a density under general coordinate transformations. Infinitesimally the supervielbein transforms as

$$\delta E_M{}^A = \xi^L \partial_L E_M{}^A + \partial_M \xi^L E_L{}^A \tag{B.19}$$

Using (B.12) one finds

$$\begin{aligned} \delta E &= \xi^L \partial_L E + E \cdot E_A{}^M \partial_M \xi^L E_L{}^A (-1)^a \\ &= \xi^L \partial_L E + E \cdot \partial_L \xi^L (-1)^e \\ &= \partial_L (\xi^L E)(-1)^e \end{aligned} \tag{B.20}$$

which is the transformation property of a superspace density. Under local Lorentz transformations E is invariant.

SUPERFORMS

We will use the Wess-Zumino <u>double</u> grading system rather than the Berezin single-grading. The advantage of the former is that the rule for applying the exterior derivative will be the same as in ordinary space (from the right) – also, expressions written in differential forms will not contain Grassmannian minus signs.

$$dz^M = (dx^m, d\theta^\mu)$$

If g denotes fermionic/bosonic parity of quantities like x^m, θ^μ defined previously and h denotes an independent parity due to the differential, then,

$$\begin{array}{lll} & & (g,h) \\ \partial_m, & x^m \ldots & (0,0) \\ \partial_\mu, & \theta^\mu \ldots & (1,0) \\ d, & dx^m \ldots & (0,1) \\ & d\theta^\mu \ldots & (1,1) \end{array}$$

The rule for how to commute quantities is then simply,

$$A_1 A_2 = (-1)^{g_1 g_2 + h_1 h_2} A_2 A_1$$

for example: $\theta^\mu dx^m = dx^m \theta^\mu$, $dx^m d\theta^\mu = -d\theta^\mu dx^m$

Important consequences of these definitions are the following:

(1) $$d^2 = (dz^M \partial_M)(dz^N \partial_N) = -(dz^N \partial_N)(dz^M \partial_M)$$
$$= 0$$

(2) $$d\, dx^m \cdot = -dx^m\, d \cdot$$
$$d\, d\theta^\mu \cdot = -d\theta^\mu\, d \cdot$$

so $$d\, dz^M \cdot = -dz^M\, d \cdot$$

(3) from (2) it follows

$$d(\omega_1 \omega_2) = \omega_1 d\omega_2 + (-1)^{p_2} d\omega_1\, \omega_2$$

where ω_2 is a p_2 form ($p_2 = h_2$)

Super vierbein, Super connection 1-forms:

$$E^A = dz^M E_M{}^A \qquad \Phi_A{}^B = dz^M \Phi_{M,A}{}^B$$

h parity $= 1$ $\qquad E^A = (E^a, E^\alpha)$

g parity $= 0, 1$

Grassmannian minus sign notation:

$$u^A v^B = (-1)^{ab} v^B u^A$$

(u, v = 0-forms) $\qquad$ lower case latin

8.1

Super-Poincaré theorem:

If ω is a form and $d\omega = 0$, then there exists (locally) α, such that $d\alpha = \omega$.

Proof: In the proof we will take ω to be a one-form (it can easily be extended)

$$\omega = dz^M \underbrace{\omega_M(z)}_{\text{0-form}}$$

$$d\omega = dz^M dz^N \partial_N \omega_M = -(-1)^{mn} dz^N dz^M \partial_N \omega_M$$

$$= -(-1)^{mn} dz^M dz^N \partial_M \omega_N$$

$$d\omega = 0 \implies \partial_N \omega_M = (-1)^{mn} \partial_M \omega_N$$

construct α,

$$\alpha(z) = \int_0^1 dt \; z^M \omega_M(zt)$$

$$d\alpha = \int_0^1 dt \left\{ \underbrace{z^M (dz^N \partial_N \omega_M)}_{mn} + dz^M \omega_M \right\}$$

$$z^M dz^N \partial_M \omega_N (-1)$$

$$dz^N z^M \partial_M \omega_N (-1)^{\cancel{mn} + \cancel{mn}}$$

Consider $z^M \partial_M f(zt)$

since $z^M \partial_M$ is commuting (in both senses) the ordinary Leibniz rule holds if we imagine f expanded out ($f = a + b\,zt + c\,zt\,zt + \cdots$) $z^M \partial_M$ just replaces every factor zt with just z ; thus

$$z^M \partial_M f(zt) = t \frac{d}{dt} f(zt)$$

so
$$d\alpha = \int_0^1 dt \left\{ t\, dz^N \frac{d}{dt} \omega_N(zt) + dz^M \omega_M(zt) \right\}$$

$$= \int_0^1 dt \frac{d}{dt} \left\{ t\, dz^N \omega_N(zt) \right\}$$

$$= dz^N \omega_N(z) = \omega$$

GEOMETRY IN SUPERSPACE

v^A , u_A zero-forms

Under local linear transformations in the tangent space:

$$\delta v^A = v^B X_B{}^A \,, \quad \delta u_A = - X_A{}^B u_B$$

note that the <u>order</u> is now important so that $\delta(v^A u_A) = 0$.

Covariant differentials are defined as:

$$Dv^A = dv^A + v^B \Phi_B{}^A \qquad (1)$$

$$DU_A = dU_A - \Phi_A{}^B U_B \qquad (2)$$

note that v and U appear on the <u>same</u> respective side of $\Phi_A{}^B$ as before for $X_A{}^B$. This insures that the transformation rule for Φ has no Grassmannian minus signs: (check!)

$$\delta \Phi_A{}^B = \Phi_A{}^C X_C{}^B - X_A{}^C \Phi_C{}^B - dX_A{}^B$$

Written in terms of the coefficient functions however, there will be minus signs:

$$D = dz^M D_M$$

coefficient of dz^M in (1):

$$D_M v^A = \partial_M v^A + (-1)^{mb} v^B \Phi_{M,B}{}^A$$

in (2): $D_M U_A = \partial_M U_A - \Phi_{M,A}{}^B U_B$

When we covariantly differentiate higher forms, there is an additional minus sign associated with <u>lower</u> indicies.

(cont.)

For example, if w_A is a p-form, then

$$D w_A = d w_A - (-1)^p \Phi_A{}^B w_B$$

This is needed if D is to have the same product rule as d. [Check that $D(u^A w_A) = d(u^A w_A)$]

The torsion and curvature are defined by:

$$T^A = DE^A = dE^A + E^B \Phi_B{}^A$$
$$= \tfrac{1}{2} dz^N dz^M T_{MN}{}^A = \tfrac{1}{2} E^C E^B T_{BC}{}^A$$

$$R_A{}^B = d\Phi_A{}^B + \Phi_A{}^C \Phi_C{}^B = \tfrac{1}{2} dz^N dz^M R_{MNA}{}^B$$
$$= \tfrac{1}{2} E^D E^C R_{CD,A}{}^B$$

From these definitions it is easy to derive the Bianchi identities:

$$DT^A = E^B R_B{}^A$$
$$DR_A{}^B = dR_A{}^B + R_A{}^C \Phi_C{}^B - \Phi_A{}^C R_C{}^B = 0$$

Note that since certain components of $R_{MNA}{}^{B}$ (namely $R_{\mu\nu A}{}^{B}$) are defined by <u>anti</u>commutators ($\{D_\mu, D_\nu\} v^A$), the other derivation of the Bianchi identities via the Jacobi identity is complicated by the appearance of mixed commutators. The use of forms takes into account those extra signs.

Later we will need the Bianchi identities with only tangent space indices on the torsion:

$$DT^D - E^C R_C{}^D = 0$$

$$\underbrace{D(E^C E^B T_{BC}{}^D)} - E^C E^B E^A R_{ABC}{}^D = 0$$

$$E^C E^B D T_{BC}{}^D + \underbrace{E^C D E^B T_{BC}{}^D - (DE^C) E^B T_{BC}{}^D}$$

$$(E^C T^B - \underbrace{T^C E^B}_{E^B T^C (-1)^{ab}}) \underbrace{T_{BC}{}^D}_{T_{CB}{}^D (-1)^{bc}}$$

$$\rightarrow 2E^C T^B T_{BC}{}^D = E^C E^B E^A T_{AB}{}^E T_{EC}{}^D$$

$$\longrightarrow E^C E^B E^A \left\{ D_A T_{BC}{}^D + T_{AB}{}^E T_{EC}{}^D - R_{ABC}{}^D \right\} = 0$$

Similarily, the other identity becomes:

$$E^C E^B E^A \{ D_A R_{BCD}{}^E + T_{AB}{}^F R_{FCD}{}^E \} = 0$$

TANGENT SPACE GROUP

The obvious generalization of an invariant numerical tensor

$$\eta_{AB} = (-1)^{ab} \eta_{BA}$$

in the tangent space does not lead to supergravity but a different theory (which does however include Einstein's theory).

Instead, one can choose a more restricted group where one requires the existence of a particular basis where the matrices $X_A{}^B$ satisfy:

$$X_a{}^\beta = X_\alpha{}^b = 0 \qquad X_a{}^b = L_a{}^b(z)$$

$$X_\alpha{}^\beta = \tfrac{1}{2} L_a{}^b (\Sigma_b{}^a)_\alpha{}^\beta$$

$L_a{}^b$ = infinitesimal Lorentz matrix

So vectors and spinors are rotated by the same Lorentz transformation which is local

(cont)

in both x <u>and</u> θ. Since the connection and curvature are matrices belonging to the algebra of the tangent space group, they satisfy the same restrictions as $X_A{}^B$. In particular:

$$R_{CD,a\beta} = R_{CD,\alpha b} = 0$$

$$R_{CD,ab} = -R_{CD,ba}$$

$$(R_{CD})_{\alpha\beta} = -(\gamma_0 R_{CD} \gamma_0)_{\beta\alpha}$$

The final restriction on the geometry involves imposing constraints on the supertorsion. Prescribing all the components would yield a "rigid" superspace. Instead, one constrains

$$T_{\alpha\beta}{}^c = 2i(\gamma^c)_{\alpha\beta} \qquad T_{\alpha\beta}{}^\gamma = 0$$

$$T_{ab}{}^c = 0 \qquad T_{\alpha b}{}^c = 0$$

but leaves $T_{\alpha b}{}^\gamma$ and $T_{ab}{}^\gamma$ undetermined. ($T_{ab}{}^\gamma$ is related to the Rarita-Schwinger field strength $\psi_{ab}{}^\gamma$)

These constraints plus the restriction of the tangent space group turn out to be sufficient to express all components of the superconnection in terms of the superviebein and its derivctives.

HOMEWORK PROBLEMS

(1) Show that $R_{\ell m,ns} = -R_{m\ell,ns} = -R_{\ell m,sn}$

and $\underset{\ell mn}{\S} R_{\ell m,ns} = 0$

imply : $R_{\ell m,ns} = R_{ns,\ell m}$

(2) Prove: $\partial_m (e v^m) = e \tilde{D}_a v^a + e v^b T_{ba}{}^a$

(3) Remove the inconsistency of charged massless vector with minimal coupling by going to Yang-Mills form.

(4) (optional) Show that (2) follows from an invariance.

(5) Show that $\int \frac{1}{4} R^{ab} e^c e^d \epsilon_{abcd} = \int -\frac{1}{2} R e \, d^4x$

(6) Find the coefficients in the Gauss-Bonnet integrand, $(\alpha R_{abcd} R^{abcd} + \beta R_{ab} R^{ab} + \gamma R^2) e$

Answer to problem (3)

covariant derivative: $D_m v = \partial_m v - [A_m, v]$

field strength: $F^{mn} = \partial_m A_n - \partial_n A_m - [A_m, A_n]$

equations of motion: $D_m F^{mn} = 0$

possible source of constraint $[D_n, D_m] F^{mn} = 0$

(this produced an algebraic constraint in the minimally coupled case)

Now $\underline{D_m F^{mn} = \partial_m F^{mn} - [A_m, F^{mn}]}$

$$D_n D_m F^{mn} = \partial_n \partial_m F^{mn} - [\partial_n A_m, F^{mn}]$$

$$- [A_m, \partial_n F^{mn}] - [A_n, \partial_m F^{mn}] +$$

$$+ [A_n, [A_m, F^{mn}]]$$

$$(D_n D_m - D_m D_n) F^{mn} = [\partial_m A_n - \partial_n A_m, F^{mn}]$$

$$\underbrace{- [A_m, \cancel{[F^{mn}, A_n]}] - [F^{mn}, [A_n, A_m]]}_{\text{for these the Jacobi identity was used on the above}} - [A_m, \cancel{[A_n, F^{mn}]}]$$

Thus: $[D_n, D_m] F^{mn} = [F_{mn}, F^{mn}] \equiv 0$

ie. the "constraint" vanishes <u>identically</u>.

Dragon's theorem.

Ordinary Bianchi identities

$$d T^a + T^b \omega_b{}^a - e^b R_b{}^a = 0 \rightarrow \mathcal{D}T = eR$$

$$d R_a{}^b + R_a{}^c \omega_c{}^b - \omega_a{}^c R_c{}^b = 0 \rightarrow \mathcal{D}R = 0$$

Take the $\mathcal{D}$ of the first

$$\mathcal{D}^2 T = e\mathcal{D}R + \mathcal{D}e\,R = e\mathcal{D}R + TR$$

but $\mathcal{D}^2 T = TR$

so the first Bianchi identity implies

$$e\mathcal{D}R = 0 \quad \text{or explicitly}$$

$$e^a \left(d R_a{}^b + R_a{}^c \omega_c{}^b - \omega_a{}^c R_c{}^b \right) = 0$$

This is not quite the second Bianchi identity because one cannot drop the e^a. Dragon has shown that in superspace with the reducible structure group, the second Bianchi identity actually follows.

Conformal Transformations $\Lambda(x)$: $\delta g_{mn} = 2\Lambda g_{mn}$

$$\delta e_m{}^a = \Lambda e_m{}^a \qquad \delta e_a{}^m = -\Lambda e_a{}^m$$

the metric connection is given by

$$2\,\omega_{m\,ab}(e) = C_{abm} - C_{bm,a} - C_{mab}$$

$$C_{mna} = \partial_m e_{na} - \partial_n e_{ma}$$

One can easily check that

$$\delta\omega_{mab}(e) = e_a{}^n(\partial_n\Lambda)\, e_{mb} - e_b{}^n(\partial_n\Lambda)\, e_{ma}$$

$$\delta\omega_{cab}(e) = -\Lambda\,\omega_{cab} + e_a{}^n\partial_n\Lambda\;\eta_{cb} - e_b{}^n\partial_n\Lambda\,\eta_{ca}$$

$$\delta\,\omega_{cab}(e)\,\varepsilon^{cabd} = -\Lambda\,\omega_{cab}(e)\,\varepsilon^{cabd}$$

$$\delta e = 4\Lambda e \qquad e = \det e_m{}^a$$

Therefore (see p. 23) the massless Majorana or Dirac action is conformally invariant if one takes $\delta\psi = -\frac{3}{2}\Lambda\psi$

The Maxwell action is conformally invariant for $\delta v_m = 0 \qquad \delta v_a = -\Lambda v_a$

How does the Riemann tensor transform under conformal transf.s?

$$\delta R_{mn\,ab} = \mathcal{D}_m \delta\omega_{n\,ab} - \mathcal{D}_n \delta\omega_{m\,ab}$$

$$\delta R_{cd\,ab} = -2\Lambda R_{cd\,ab} + \underbrace{e_c{}^m e_d{}^n \delta R_{mn\,ab}}$$

$$e_d{}^n \mathcal{D}_c \delta\omega_{n\,ab} - e_c{}^n \mathcal{D}_d \delta\omega_{n\,ab} \quad \text{(no torsion)}$$

$$= \mathcal{D}_c\left(e_d{}^n \delta\omega_{n\,ab}\right) - \mathcal{D}_d\left(e_c{}^n \delta\omega_{n\,ab}\right)$$

$$= \mathcal{D}_c\left(\mathcal{D}_a\Lambda\, \eta_{db} - \mathcal{D}_b\Lambda\, \eta_{da}\right) - \mathcal{D}_d\left(\mathcal{D}_a\Lambda\, \eta_{cb} - \mathcal{D}_b\Lambda\, \eta_{ca}\right)$$

$$\delta R_{cd\,ab} = -2\Lambda R_{cd\,ab} + \mathcal{D}_c\mathcal{D}_a\Lambda\, \eta_{db} - \mathcal{D}_c\mathcal{D}_b\Lambda\, \eta_{da} - \mathcal{D}_d\mathcal{D}_a\Lambda\, \eta_{cb} + \mathcal{D}_d\mathcal{D}_b\Lambda\, \eta_{ca}$$

For the Ricci tensor one has

$$\delta R_{ca} = -2\Lambda R_{ca} + 2\mathcal{D}_c\mathcal{D}_a\Lambda + \mathcal{D}_b\mathcal{D}^b\Lambda\, \eta_{ca}$$

Also

$$\delta R = -2\Lambda R + 6\mathcal{D}_a\mathcal{D}^a\Lambda$$

The gravity action is not conformally invariant, obviously.

The Weyl conformal tensor is defined as

$$C_{cdab} = R_{cdab} - \frac{1}{2}\left(R_{ca}\eta_{db} - R_{cb}\eta_{da} - R_{da}\eta_{cb} + R_{db}\eta_{ca}\right) + \frac{1}{6}R\left(\eta_{ca}\eta_{db} - \eta_{cb}\eta_{da}\right)$$

It transforms simply under conformal transf.s

$$\delta C_{cdab} = -2\Lambda\, C_{cdab}$$

and is traceless

$$C_{cda}{}^{d} = 0$$

(In two-component spinor notation the Weyl tensor becomes a totally symmetric spinor $C_{\alpha\beta\gamma\delta}$ plus its compl. conj.)

This tensor characterizes spaces which can be transformed into each other by a conformal transformation. If the Weyl tensor vanishes the space is "conformally flat", i.e. it can be transformed into a flat space by a conformal transf.

The Weyl Lagrangian of "conformal gravity"

$$C_{cdab}\, C^{cdab}\, e$$

is conformally invariant, but is afflicted with ghosts (equations of fourth order in the derivatives)

The action for a scalar in curved space

$$I_1 = -\frac{1}{2}\int g^{mn}\partial_m A\,\partial_n A\sqrt{g} = -\frac{1}{2}\int \mathcal{D}_a A\,\mathcal{D}^a A\, e$$

is not conformally invariant. If one transforms the scalar as

$$\delta A = -\Lambda A$$

one finds

$$\underbrace{e\,(\underset{+4\Lambda}{e}\,\underset{-\Lambda}{e_a{}^m}\,\partial_m A)(\underset{-\Lambda}{e_b{}^n}\,\partial_n A)\eta^{ab}}$$

$$\delta I_1 = -\frac{1}{2}\int\left[2\mathcal{D}_a A\,\mathcal{D}^a(-\Lambda A)e + 2\Lambda e\,\mathcal{D}_a A\,\mathcal{D}^a A\right]$$

$$= -\frac{1}{2}\int\left(-2\,\mathcal{D}_a A\,\mathcal{D}^a\Lambda\, A\, e\right) = -\frac{1}{2}\int(\mathcal{D}_a\mathcal{D}^a\Lambda)A^2 e$$

However, observe that

$$I_2 = \frac{1}{12}\int A^2 R e$$

transforms as

$$\delta I_2 = \frac{1}{2}\int A^2(\mathcal{D}_a\mathcal{D}^a\Lambda)e\,.$$

Therefore $I_1 + I_2$ is conformally invariant. The energy momentum tensor obtained from this action by varying with respect of the metric or the vierbein is the Traceless "improved" energy momentum tensor.

PFAFFIANS

M = real, antisymmetric, $2n \times 2n$ matrix

We define the Pfaffian of M as:

$$P(M) = \frac{1}{2^n n!} M_{k_1 l_1} \cdots M_{k_n l_n} \epsilon^{k_1 \cdots k_n l_1 \cdots l_n}$$

$$\epsilon^{1\,2 \cdots 2n-1\; 2n} = +1$$

a useful property is:

$$P(M) = (\det M)^{1/2} .$$

proof: compute the following c-number in two ways,

$$\frac{\partial}{\partial \eta^1} \cdots \frac{\partial}{\partial \eta^{2n}} (\eta^k M_{kl} \eta^l)^n$$

$\eta^1, \ldots, \eta^{2n}$ are <u>Grassmannian</u>.

(1) first way:

$$\partial_1 \cdots \partial_{2n} \underbrace{\eta^{k_1} \eta^{l_1} \cdots \eta^{k_n} \eta^{l_n}}_{\pm \epsilon^{k_1 \cdots k_n l_1 \cdots l_n} \eta^1 \cdots \eta^{2n}} M_{k_1 l_1} \cdots M_{k_n l_n}$$

$$= \pm 2^n n!\, P(M)$$

(2) second way:

$$M = N^T J N \qquad J = \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right)$$

$$N^k{}_\ell \, \eta^\ell = \xi^k \qquad \frac{\partial}{\partial \eta^\ell} = \frac{\partial \xi^k}{\partial \eta^\ell} \frac{\partial}{\partial \xi^k}$$

$$= N^k{}_\ell \frac{\partial}{\partial \xi^k}$$

$$\frac{\partial}{\partial \eta^1} \cdots \frac{\partial}{\partial \eta^{2n}} = N^{k_1}{}_1 N^{k_2}{}_2 \cdots N^{k_{2n}}{}_{2n} \frac{\partial}{\partial \xi^{k_1}} \cdots \frac{\partial}{\partial \xi^{k_{2n}}}$$

$$= \pm (\det N) \frac{\partial}{\partial \xi^1} \cdots \frac{\partial}{\partial \xi^{2n}}$$

$$(\eta^k M_{k\ell} \eta^\ell)^n = (\xi^k J_{k\ell} \xi^\ell)^n$$

$$= (2\xi^1\xi^{n+1} + 2\xi^2\xi^{n+2} + \cdots + 2\xi^n\xi^{2n})^n$$

$$= 2^n n! (\pm \xi^1 \xi^2 \cdots \xi^{2n})$$

so $\quad \frac{\partial}{\partial \eta^1} \cdots \frac{\partial}{\partial \eta^{2n}} (\eta^k M_{k\ell} \eta^\ell)^n = \pm 2^n n! (\det N)$

Since $\quad \det M = (\det N)^2 \underbrace{\det J}_{= (-1)^n \cdot (-1)^n = +1}$

$$\det N = \pm (\det M)^{1/2}$$

thus, combining (1) and (2), $P(M) = \pm (\det M)^{1/2}$

to get the sign, note that $P(\ )$ and $\det(\ )$ are continuous functions of the M_{kl}'s; thus it is enough to establish only one case,

$$M = J : \qquad \det M = +1 .$$

$$P(M) = \frac{1}{2^n n!}\left\{ (+1)^n \epsilon^{12\cdots n\, n+1 \cdots 2n} + \cdots \right\}$$

$$= +1 \qquad \text{QED}$$

Now suppose we have n antisymmetric matrices $\alpha, \beta, \gamma, \ldots$. Define:

$$(\alpha, \beta, \gamma, \ldots) = \frac{1}{2^n n!}\, \alpha_{k_1 l_1} \beta_{k_2 l_2} \gamma_{k_3 l_3} \cdots \epsilon^{k_1 k_2 k_3 \cdots l_1 l_2 l_3 \cdots}$$

let $r, s, u, \ldots$ be n parameters, then,

$$P(r\alpha + s\beta + u\gamma + \cdots) = r^n P(\alpha) + \cdots +$$

$$+ (rsu\cdots)(n!)(\alpha, \beta, \gamma, \ldots) + \cdots$$

(We're only interested in the term linear in $r, s, u, \ldots$)

---

let ρ be another antisymmetric matrix, then,

$$M(t) = e^{t\rho} M e^{-t\rho} \text{ is antisymmetric}$$

Moreover, since $P(\ )$ is the square root of the determinant, it follows that

$$P(M(t)) = P(M)$$

In particular,

$$(\alpha(t), \beta(t), \gamma(t), \dots) = (\alpha, \beta, \gamma, \dots)$$

Take $\frac{d}{dt}$ of both sides and set $t=0$:

$$\underbrace{([\rho,\alpha], \beta, \gamma, \dots) + (\alpha, [\rho,\beta], \gamma, \dots) + \dots}_{n \text{ terms}} = 0$$

The other properties of $(\alpha, \beta, \gamma, \dots)$, ie. multilinearity and symmetry, follow trivially from the definition.

Specialize to the case $n=2$ (α, etc. belong to algebra of Lorentz group)

$$(\alpha, \beta) = \frac{1}{8} \alpha_{ab} \beta_{cd} \epsilon^{acbd} = \frac{1}{4} \mathrm{tr}(\tilde{\alpha} \beta)$$

$$(\tilde{\alpha})^{cd} = \frac{1}{2} \alpha_{ab} \epsilon^{abcd} \quad \text{(dual)}$$

Since $(\beta\gamma)^T = (-\gamma)(-\beta) = \gamma\beta$, $\beta\gamma + \gamma\beta$ is <u>symmetric</u>

Thus $(\alpha, \beta\gamma + \gamma\beta) = 0$

Combine this with the invariance property above with $\rho = \gamma$,

$$(\gamma\alpha - \alpha\gamma, \beta) + (\alpha, \gamma\beta - \beta\gamma) = 0$$

to give:

$$(\gamma\alpha, \beta) + (\alpha, \gamma\beta) = 0$$
$$(\alpha\gamma, \beta) + (\alpha, \beta\gamma) = 0$$
$$(\gamma\alpha, \beta) - (\alpha, \beta\gamma) = 0$$

In summary, the basic properties of the invariant $(\alpha, \beta) = \frac{1}{4} \operatorname{tr} \tilde{\alpha}\beta$ are:

(1) bilinearity

(2) $(\alpha, \beta) = (\beta, \alpha)$

(3) $(\gamma\alpha, \beta) = -(\alpha, \gamma\beta)$
$(\alpha\gamma, \beta) = -(\alpha, \beta\gamma)$
$(\gamma\alpha, \beta) = (\alpha, \beta\gamma)$

There are additional minus signs if the matrices have anticommuting elements. For example, if $\omega_1, \omega_2, \omega_3$ are antisymmetric matrix valued <u>one forms</u>, then

$$(\omega_1, \omega_2) = -(\omega_2, \omega_1) \quad (\omega_1 \omega_2, \omega_3) = (\omega_2, \omega_1 \omega_3)$$

PONTRYAGIN INDEX

This is a general "topological invariant"; it applies in the case of any tangent space group.

$$I_p \propto \int \underbrace{\mathrm{tr}\, F^2}_{\text{4 form}} \propto \int d^4x\, \epsilon^{lmnr} F_{lm,a}{}^{b} F_{nr,b}{}^{a}$$

Locally, the integrand may be written as

$$\mathrm{tr}\, F^2 = d\left\{ \mathrm{tr}\left(dA A + \tfrac{2}{3} A^3\right)\right\}$$

Check: $F = dA + A^2$

$$\mathrm{tr}\, F^2 = \mathrm{tr}\left\{(dA)^2 + dA A^2 + A^2 dA + A^4\right\}$$

now $\mathrm{tr}\, A^2 dA = \mathrm{tr}\, dA A^2$

$$\mathrm{tr}\, A^4 = \mathrm{tr}\, \underset{\text{1 form}}{A}\, \underset{\text{3 form}}{A^3} = -\mathrm{tr}\, A^3 A = -\mathrm{tr}\, A^4 = 0$$

So $\mathrm{tr}\, F^2 = \mathrm{tr}\left\{(dA)^2 + 2\, dA A^2\right\}$

whereas, $d\left\{\mathrm{tr}\left(dA A + \tfrac{2}{3} A^3\right)\right\} =$

$$\mathrm{tr}\left\{(dA)^2 + \tfrac{2}{3} A^2 dA - \tfrac{2}{3} A dA A + \tfrac{2}{3} dA A^2\right\}$$

$$= \mathrm{tr}\left\{(dA)^2 + 2\, dA A^2\right\}$$

Since the integrand is a differential (of a gauge <u>dependent</u> quantity) this shows, in the case of flat space, that I_p depends only on the asymptotic behavior of the fields. To show it truly is an invariant consider infinitesimal (local) variations of the potential,

$$A \to A + \delta A$$

δA is gauge invariant (in the sense of F) since,

$$\delta_{gauge}(A_m) = \partial_m X - [A_m, X]$$

$$\delta_{gauge}(A_m + \delta A_m) = \partial_m X - [A_m + \delta A_m, X]$$

$$\delta_{gauge}(\delta A_m) = -[\delta A_m, X]$$

As shown on page 29,

$$\delta F = D\,\delta A = d\,\delta A + A\,\delta A + \delta A\, A$$

(Note: $DF = dF + FA - AF$, difference in signs is that mentioned on page 84
$F =$ 2 form, $\delta A =$ one form)

$$\delta\, \mathrm{tr}\, F^2 = 2\,\mathrm{tr}\, F\,\delta F = 2\,\mathrm{tr}\, F\, D\,\delta A$$

(Cont.)

$$= 2\,\mathrm{tr}\, D(F\,\delta A) \qquad \left(DF = 0 \quad \text{Bianchi identity}\right)$$

$$= d\,\underbrace{\{2\,\mathrm{tr}\, F\,\delta A\}}_{\text{gauge invariant}}$$

Thus, if I_p is some value for a particular potential, then it will remain constant when the potential is varied infinitesimally

---

All this applies to the Riemann tensor,

$$R = d\omega + \omega^2 \,, \qquad DR = 0$$

$$\mathrm{tr}\, R^2 = d\left\{\mathrm{tr}\left(d\omega\,\omega + \tfrac{2}{3}\omega^3\right)\right\}$$

$$\delta\,\mathrm{tr}\, R^2 = d\left\{2\,\mathrm{tr}\, R\,\delta\omega\right\}$$

Since the integrand $\mathrm{tr}\, R^2$ is a pseudo-scalar density, such a quantity could never occur in the effective action of quantum gravity and hence is not useful in discussing divergence problems.

GAUSS-BONNET

When the tangent space group is the orthogonal group (as in ordinary gravity) we can construct another topological invariant using the Pfaffian formalism introduced before. This is the "Euler-characteristic" (= # of handles for 2-manifold); the Gauss-Bonnet formula gives this number in terms of the geometry:

$$I_E \propto \int \underbrace{(R,R)}_{\text{4-form}} \propto \int d^4x\, \epsilon^{lmnr} R_{lm,ab} R_{nr,cd} \cdot \epsilon^{abcd}$$

This construction is appropriate since R is an antisymmetric matrix (valued 2-form). The rest of the discussion is very similar to that for the Pontryagin index:

$$R = d\omega + \omega^2$$

$$(R,R) = (d\omega, d\omega) + \underbrace{(d\omega, \omega^2) + (\omega^2, d\omega)}_{2(d\omega,\omega^2)} + (\omega^2, \omega^2)$$

$$(\omega^2, \omega^2) = \left\{ \begin{array}{l} (\omega\omega, \omega^2) = +(\omega, \omega\omega^2) \\ (\omega\omega, \omega^2) = -(\omega, \omega^2\omega) \end{array} \right\} = 0$$

On the other hand, consider

$$d\left\{\left(d\omega + \tfrac{2}{3}\omega^2, \omega\right)\right\}$$

$$= (d\omega, d\omega) + \tfrac{2}{3}\Big\{\underbrace{-(\omega d\omega, \omega)}_{+(d\omega,\omega^2)} + \underbrace{(d\omega\,\omega, \omega)}_{+(d\omega,\omega^2)} + \underbrace{(\omega^2, d\omega)}_{+(d\omega,\omega^2)}\Big\}$$

$$= (d\omega, d\omega + 2\omega^2)$$

Conclusion:

$$(R,R) = d\left\{\left(d\omega + \tfrac{2}{3}\omega^2, \omega\right)\right\}$$

The check of invariance under infinitesimal deformations of the potential is also nearly identical:

$$\delta(R,R) = 2(R, \delta R) = 2(R, D\delta\omega)$$

$$= -2(\underbrace{DR}_{0}, \delta\omega) + 2(R, D\delta\omega) = d\,2(R, \delta\omega)$$

(R,R) is a <u>scalar</u> density; in fact,

$$(R,R) \propto d^4x\, e\left\{R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2\right\}$$

The Pontryagin / Gauss Bonnet discussions have been so similar because these are just two examples of a general mathematical object called an "invariant, symmetric, polynomial"

$P(A, B, C, \dots)$ symmetric w.r.t. interchange of any pair of matrices (a-forms).

and,

$$P([\rho, A], B, C, \dots) + P(A, [\rho, B], C, \dots) + \dots = 0$$

Pontryagin : $P(A,B) = \operatorname{tr} AB$

Gauss-Bonnet : $P(A,B) = \frac{1}{4} \operatorname{tr} \tilde{A} B$ A, B <u>antisymmetric</u>

The invariance relation above is fundamental; relations such as

$$(\gamma\alpha, \beta) + (\alpha, \gamma\beta) = 0$$

only obscure the picture and are not needed. As an example, take $\rho, A = 1$ forms, $B =$ 2 form. Taking into account minus signs due to forms,

$$P(\{\rho, A\}, B) - P(A, [\rho, B]) = 0$$

Now take $\rho = A = \omega$, $B = \omega^2$:

$$(\omega^2, \omega^2) = \tfrac{1}{2}(\{\omega, \omega\}, \omega^2\} = \tfrac{1}{2}(\omega, \underbrace{[\omega, \omega^2]}_{0}) = 0$$

drop "P" as in earlier notation.

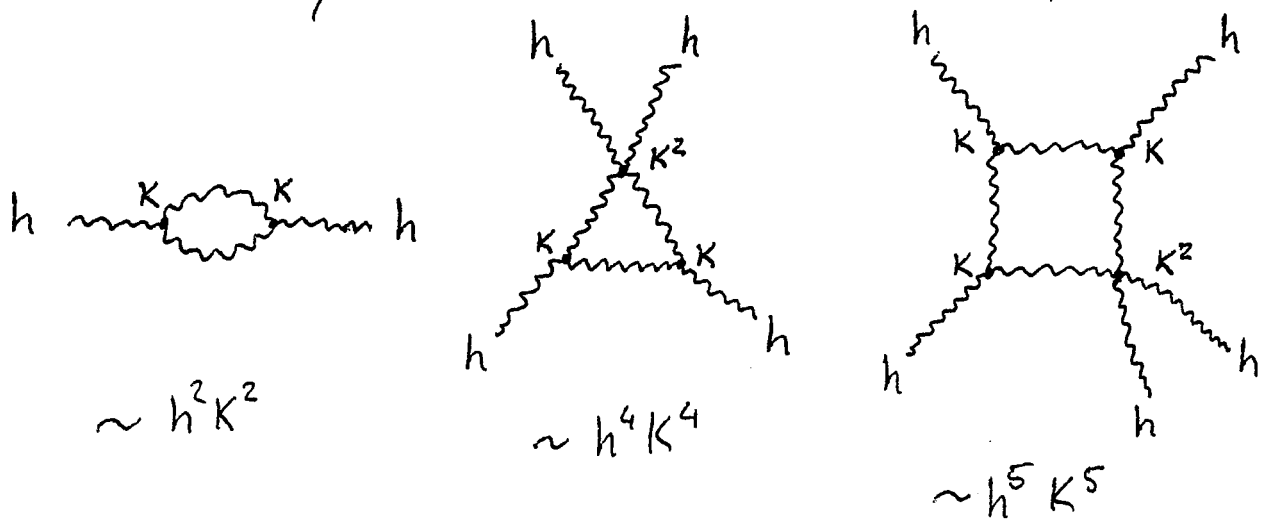
DIVERGENCES IN ORDINARY GRAVITY

We expand the Einstein action $\int d^4x \left(-\frac{1}{2K^2}\sqrt{g}\,R\right)$ in powers of K where

$$g_{mn} = \eta_{mn} + K h_{mn}$$

The first (quadratic) term is independent of K so propagators carry no power of K. In general, m-point vertices carry K^{m-2}.

When we compute the effective action to say, the "one-loop level", we essentially sum an infinite set of diagrams, all having one loop but an arbitrary number of external lines,



In this case, the total power of K equals the number of external lines. Since the sum of these must be some <u>covariant</u> combination of h_{mn}'s and

$g_{mn} \sim K h_{mn}$ we see that the sum will contain no explicit power of K (everything expressed only in terms of g_{mn}) In general, suppose a diagram has,

L loops

E external lines

V_m m point verticies

The excess power of K will be,

$$p = \sum_{m=3} (m-2) V_m - E$$

The external lines may be joined at a single vertex to complete a polyhedron having,

$$v = \sum_{m=3} V_m + 1 \quad \text{vertices}$$

$$e = \sum_{m=3} \frac{m}{2} V_m + \frac{1}{2} E \quad \text{edges}$$

$$f = L + E \quad \text{faces}$$

Using the Euler formula we have,

$$2 = v - e + f = \sum V_m + 1 - \frac{1}{2} \sum m V_m - \frac{1}{2} E + L + E$$

or $p = 2L - 2$ So the diagram will be a covariant combination of g_{mn} times K^{2L-2}.

Using <u>dimensional</u> regularization, the general expression for the divergent part of the effective action is,

$$S^{div.}(L\text{-loops}) = \frac{\kappa^{2L-2}}{(d-4)^L} \int d^4x \sqrt{g}\, A$$

(d = spacetime dimension)

Since the action is dimensionless, A has dimension $2L+2$ and is a covariant combination of $L+1-k$ Riemann tensors and $2k$ covariant derivatives. (A does not contain κ explicitly nor can it have R^{-1} or $(D_m D^m)^{-1}$ since these are never generated by perturbation theory.)

ABSENCE OF ONE LOOP COUNTERTERM IN ORDINARY GRAVITY

When $L=1$ there are only three terms that might appear in A,

$$R_{abcd}R^{abcd}, \quad R_{ab}R^{ab}, \quad R^2$$

(Terms involving $D_m D^m R$ or $D_m D_n R^{mn}$ are zero by partial integration.)

In perturbation theory, the background flat spacetime ($g_{mn} = \eta_{mn}$) has Euler characteristic zero. Thus, by the Gauss Bonnet formula, the term

$$R_{abcd} R^{abcd}$$

can be replaced in the effective action by a linear combination of the other two. Therefore, on the "mass-shell" with $R_{ab} = 0$, any possible counterterm vanishes. This result has also been checked by explicit term by term cancellation in the perturbation expansion.

SOLUTION OF THE CONSTRAINTS

Using the Bianchi identities (3.12) and (3.13) one can show that (3.14) and (3.16) imply that all components of the supercurvature and supertorsion are expressible in terms of the three superfields $G_{\alpha\dot\beta}$, $W_{\alpha\beta\gamma}$, R and their conjugates. Here we have used two-component spinor notation (see Appendix A). $G_{\alpha\dot\beta}$ is hermitean, $W_{\alpha\beta\gamma}$ totally symmetric. For instance

$$
\begin{aligned}
R_{\alpha\beta,\gamma\delta} &= -4\left(\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta} + \varepsilon_{\alpha\delta}\varepsilon_{\beta\gamma}\right) R^* \\
T_{\alpha,b,\gamma} \equiv T_{\alpha,\beta\dot\beta,\gamma} &= \frac{i}{4}\left(\varepsilon_{\beta\gamma} G_{\alpha\dot\beta} - 3\,\varepsilon_{\alpha\gamma} G_{\beta\dot\beta} - 3\,\varepsilon_{\alpha\beta} G_{\gamma\dot\beta}\right) \\
T_{\alpha,b,\dot\gamma} \equiv T_{\alpha,\beta\dot\beta,\dot\gamma} &= -2i\,\varepsilon_{\alpha\beta}\,\varepsilon_{\dot\beta\dot\gamma}\, R^*
\end{aligned}
\tag{3.17}
$$

$$
\oint_{ABC}\left(\mathcal{D}_A T_{BC}{}^{D} + T_{AB}{}^{F} \cdot T_{FC}{}^{D} - R_{ABC}{}^{D}\right) = 0 \tag{3.12}
$$

$$
\oint_{ABC}\left(\mathcal{D}_A R_{BCD}{}^{F} + T_{AB}{}^{G} \cdot R_{GCD}{}^{F}\right) = 0 \tag{3.13}
$$

$$
\begin{aligned}
R_{CD,ab} &= -R_{CD,ba} \\
R_{CD,\alpha\beta} &= \frac{1}{2} R_{CD,ab}\left(\Sigma^{ab}\right)_{\alpha\beta} \\
R_{CD,a\beta} &= R_{CD,\alpha b} = 0
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
T_{\alpha\beta}{}^{c} &= 2i(\gamma^{c})_{\alpha\dot\beta}, \qquad T_{\alpha\beta}{}^{\gamma} = 0 \\
T_{ab}{}^{c} &= 0, \qquad T_{\alpha b}{}^{c} = 0
\end{aligned}
\tag{3.16}
$$

$$T_{ab\gamma} = T_{\alpha\dot\alpha,\beta\dot\beta,\gamma} = -2\varepsilon_{\dot\alpha\dot\beta} W_{\alpha\beta\gamma} + \frac{1}{2}\varepsilon_{\alpha\beta}\left(\mathcal{D}_{\dot\alpha} G_{\gamma\dot\beta} + \mathcal{D}_{\dot\beta} G_{\gamma\dot\alpha}\right) + \frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}\left(\varepsilon_{\beta\gamma}\mathcal{D}_\alpha R + \varepsilon_{\alpha\gamma}\mathcal{D}_\beta R\right)$$

The remaining content of the Bianchi identities is expressed by the differential relations

$$\mathcal{D}^\alpha W_{\alpha\beta\gamma} = \mathcal{D}_\beta{}^{\dot\varepsilon} G_{\gamma\dot\varepsilon} + \mathcal{D}_\gamma{}^{\dot\varepsilon} G_{\delta\dot\varepsilon} \quad , \qquad \mathcal{D}^\alpha G_{\alpha\dot\beta} = \mathcal{D}_{\dot\beta} R^*$$

$$\mathcal{D}_{\dot\varepsilon} W_{\alpha\beta\gamma} = 0 \qquad\qquad \mathcal{D}_{\dot\varepsilon} R = 0 \tag{3.18}$$

$$\mathcal{D}_{\beta\dot\varepsilon} \equiv (\sigma^a)_{\beta\dot\varepsilon}\, \mathcal{D}_a$$

The superfield $G_{\alpha\dot\alpha}$ has a simple physical meaning, which can be exhibited by considering the coefficients of its expansion in θ . For instance, at the $\theta\bar\theta$ level one finds a tensor which contains the Einstein tensor, at the $\theta\theta\bar\theta$ level a spinor which is the Rarita - Schwinger operator (left-hand side of the Rarita-Schwinger equation). Similarly, R contains the contracted (scalar) curvature tensor, at the $\theta\theta$ level. $W_{\alpha\beta\gamma}$ contains the Weyl conformal spinor, at the θ level ; for $\theta = 0$, it contains the Rarita Schwinger field strength. Observe that some components of R are obtained from those of $G_{\alpha\dot\alpha}$ by tracing over certain indices : in superspace this is expressed by the differential identities (3.18) in θ .

In a general affine superspace one has the identity

$$\partial_M\left(E v^A E_A{}^M\right)(-1)^a = E\left(\mathcal{D}_A v^A + v^B T_{BA}{}^A\right)(-1)^a \tag{3.19}$$

where v^A is an arbitrary vector and

$$E = \det E_M{}^A \tag{3.20}$$

is the graded determinant of the supervielbein matrix, which is a density in superspace . With our constraints (3.16) on the torsion, the last term $T_{BA}{}^A(-1)^a$ vanishes (some torsion components vanish and others cancel in the trace) and one has simply

$$\partial_M\left(E v^A E_A{}^M\right)(-1)^a = E\, \mathcal{D}_A v^A (-1)^a \tag{3.21}$$

Integrating over all superspace, and assuming that there are no boundary terms, one obtains

$$\int dx\, d\theta\; E\, \mathcal{D}_A v^A (-1)^a = 0 \tag{3.22}$$

a very useful formula which permits systematic use of integration by parts

The above constraints are naturally invariant under general coordinate transformations in superspace and local Lorentz transformations. They are also invariant under certain conformal transformations. In infinitesimal form these transformations involve an infinitesimal parameter function Σ which is covariantly chiral, i.e. satisfies

$$\mathcal{D}_{\dot\alpha} \Sigma = 0 \tag{3.23}$$

They are

$$\delta E_M{}^A = (\Sigma + \Sigma^*)\, E_M{}^A$$

$$\delta E_M{}^\alpha = (2\Sigma^* - \Sigma) E_M{}^\alpha - \frac{i}{2} E_M{}^a (\sigma_a)^{\alpha\dot\alpha} \mathcal{D}_{\dot\alpha} \Sigma^* \tag{3.24}$$

$$\delta E_M{}^{\dot\alpha} = (2\Sigma - \Sigma^*) E_M{}^{\dot\alpha} + \frac{i}{2} E_M{}^a (\sigma_a)^{\alpha\dot\alpha} \mathcal{D}_\alpha \Sigma$$

and

$$\delta \Omega_{M,\alpha\beta} = E_{M\alpha} \mathcal{D}_\beta \Sigma + E_{M\beta} \mathcal{D}_\alpha \Sigma + (\sigma^{ab})_{\alpha\beta} E_{Ma} \mathcal{D}_b (\Sigma + \Sigma^*) \tag{3.25}$$

For the determinant of the supervielbein, (3.24) imply

$$\delta E = 2(\Sigma + \Sigma^*) E \tag{3.26}$$

while curvature and torsion behave as given by

$$\delta R = (2\Sigma^* - 4\Sigma) R - \frac{1}{4} \mathcal{D}_{\dot\alpha} \mathcal{D}^{\dot\alpha} \Sigma^*$$

$$\delta G_{\alpha\dot\alpha} = -(\Sigma + \Sigma^*) G_{\alpha\dot\alpha} + i \mathcal{D}_{\alpha\dot\alpha} (\Sigma^* - \Sigma)$$

$$\delta W_{\alpha\beta\gamma} = -3 \Sigma \cdot W_{\alpha\beta\gamma} \tag{3.27}$$

Since the constraints are invariant under these transformations, we can say that the geometry (kinematics) is invariant under conformal transformations. The dynamics may of may not be, depending on the choice of Lagrangian. For instance, the action of supergravity, given in the next section, is only invariant under general coordinate transformations in superspace and under local Lorentz transformations, but not under conformal transformations. The action of conformal supergravity, also given below, is instead also conformally invariant.

Observe that the commutation relations together with the solution (3.17) of the Bianchi identities (plus the analogous expressions for the other components of the curvature and torsion), gives the commutator or anticommutator of any two covariant derivatives in terms of the superfields $G_{\alpha\dot\beta}$, $W_{\alpha\beta\gamma}$, R and their conjugates. Using these commutation relations it is easy to verify that the expression

$$\phi = (\mathcal{D}_{\dot\alpha} \mathcal{D}^{\dot\alpha} - 8R) \mathcal{U} \tag{3.28}$$

satisfies identically, for any U, the chirality constraint

$$\mathcal{D}_{\dot\alpha} \phi = 0 \tag{3.29}$$

Viceversa, any chiral superfield can be written in the form (3.28). More generally, if $\mathcal{U}_{\alpha\beta\gamma\ldots}$ has only undotted indices,

$$\phi_{\alpha\beta\gamma\ldots} = (\mathcal{D}_{\dot\alpha} \mathcal{D}^{\dot\alpha} - 8R) \mathcal{U}_{\alpha\beta\gamma\ldots} \tag{3.30}$$

implies

$$\mathcal{D}_{\dot\alpha} \phi_{\alpha\beta\gamma\ldots} = 0 \tag{3.31}$$

and viceversa. One also has the identity

$$\left(\mathcal{D}_{\dot\alpha}\mathcal{D}^{\dot\alpha} - 8\mathcal{R}\right)\mathcal{D}_{\dot\beta}\Lambda^{\dot\beta} = 0 \tag{3.32}$$

These relations are generalizations to curved superspace of well known identities which, in flat superspace, follow from (3.6)

4. DYNAMICS IN SUPERSPACE

In the previous section we have completely specified the geometry of superspace by the tangent group restrictions (3.14) and (3.15) and by the torsion constraints (3.16). We refer to this specification of the geometry as kinematics : it is the analogue of the specification of the geometry in Einstein's theory as being Riemannian. Just as in Einstein's theory the dynamics is given by giving an action (or the field equations), we shall now specify the dynamics by giving the action for supergravity in superspace. It is simply

$$I = \int dx\, d\theta \;\; \det E_M{}^A \tag{4.1}$$

the superspace integral of the simplest density, the determinant of the supervielbein. The action (4.1) must be varied keeping the kinematical constraints satisfied.

SUPERSPACE DENSITY FOR CHIRAL FIELDS

A general scalar in superspace, upon multiplication by the basic density (3.20), the determinant of the supervielbein, gives rise to a density which can be used as a Lagrangian. If the scalar is chiral, however, one has to proceed differently. In flat superspace a chiral superfield ϕ can be written in the form $\phi = D_{\dot\alpha}D^{\dot\alpha}U$. One can use ϕ as a Lagrangian by integrating only over $\theta^{\dot\alpha}$ (and x) but not over θ^{α}. Alternatively, one can integrate U over all four θ components (and x) with the same result. The first procedure has no analogue in curved superspace in a general gauge, the second does. Any chiral superfield ϕ, (3.29), can be written in the form (3.28), where U has a certain arbitrarity, given by (3.30). The action obtained from U can be transformed as follows

$$\int E\,\mathcal{U} \equiv -\frac{1}{8}\int \frac{E}{R}\left(\phi - \mathcal{D}_{\dot\alpha}\mathcal{D}^{\dot\alpha}\mathcal{U}\right) = -\frac{1}{8}\int \frac{E}{R}\,\phi \tag{4.13}$$

where the last equality follows by partial integration because R is chiral. We see that, in order to construct a Lagrangian density from a chiral superfield, we must multiply it by -E/8R, not by E. As an example, consider the chiral superfield $W_{\alpha\beta\gamma} W^{\alpha\beta\gamma}$. The corresponding action is proportional to

$$\int \frac{E}{R} W_{\alpha\beta\gamma} W^{\alpha\beta\gamma} \tag{4.14}$$

It is easy to verify that it is invariant under the conformal transformations (3.26) and (3.27). Indeed, (4.14) is the superspace form of the action for conformal supergravity. Observe that the action of non conformal supergravity (4.1) can be interpreted in two equivalent ways. One can think of it as being obtained by multiplying the identity superfield by E of by multiplying the chiral superfield R by E/R.

5. COUPLINGS TO SUPERSYMMETRIC MATTER

We first describe the coupling to the vector-spinor multiplet. In analogy with flat superspace, this matter supermultiplet is described by a real scalar superfield V which can undergo gauge transformations of the form

$$V \rightarrow V + i\Lambda - i\Lambda^*, \qquad \mathcal{D}_{\dot\alpha}\Lambda = 0 \tag{5.1}$$

The superfield

$$W_\alpha = \left(\mathcal{D}_{\dot\sigma}\mathcal{D}^{\dot\sigma} - 8R\right)\mathcal{D}_\alpha V$$

is gauge invariant under (5.1). This can be verified by using the commutation relations among the covariant derivatives. From (3.30) and (3.31) we see that it is chiral

$$\mathcal{D}_{\dot\alpha} W_\alpha = 0 \tag{5.2}$$

It also satisfies identically

$$\mathcal{D}^\alpha W_\alpha = \mathcal{D}_{\dot\alpha} W^{\dot\alpha} \tag{5.3}$$

which is the supersymmetric form of the electromagnetic Bianchi identities. The gauge invariant superfield W_α gives the spin 1/2 field of the supermultiplet for $\theta = 0$ and contains the electromagnetic field strength at the θ level. We can take as action, up to a proportionality constant,

$$\int E \left(\frac{1}{R} W^\alpha W_\alpha + \frac{1}{R^*} W_{\dot\alpha} W^{\dot\alpha} \right) \tag{5.4}$$

(observe that we use the density E/R appropriate to a chiral Lagrangian). The corresponding equation of motion is

$$\mathcal{D}^\alpha W_\alpha + \mathcal{D}_{\dot\alpha} W^{\dot\alpha} = 0 \tag{5.5}$$

which, combined with (5.3), gives

$$\mathcal{D}^\alpha W_\alpha = \mathcal{D}_{\dot\alpha} W^{\dot\alpha} = 0 \tag{5.6}$$

The action (5.4) is invariant under the gauge transformation (5.1). It is also invariant under conformal transformations provided one assumes the conformal property

$$\delta V = 0 \tag{5.7}$$

which implies

$$\delta W_\alpha = -3 \Sigma W_\alpha \tag{5.8}$$

This conformal invariance can be easily verified by combining (5.8) with (3.26) and (3.27) and integrating by parts.

If one adds the actions (4.1) and (5.4), one has the superspace formulation of the theory given first in component form. Conformal invariance is spoiled by the supergravity term. An interesting alternative is obtained by adding to (5.4) the action

$$\int E\, e^{2V} \tag{5.9}$$

Expanding the exponential, the first term gives the supergravity action and the second term (linear in V) is a generalization of the so called Fayet-Illiopoulos terms, which in rigid supersymmetry gives rise to spontaneous breaking of supersymmetry . The action (5.9) is neither gauge invariant (under (5.1)) nor conformal invariant (under (3.26), (5.7)) but is invariant under the combination of a gauge transformation and a conformal transformation provided their parameters (which are both chiral) are related

$$\Sigma + i\Lambda = 0 \tag{5.10}$$

Since (5.4) is both gauge and conformal invariant one can add it to (5.9) and one obtains the curved superspace version of the Fayet-Illiopoulos term. It is gauge invariant, if we define as gauge transformation the combination specified by (5.10). Using this gauge invariance one can partially choose the gauge (so called Wess-Zumino gauge) so that the gauge dependent fields of the vector spinor supermultiplet (except for the vector field) vanish. One is left with an ordinary gauge invariance for the vector field and the theory takes the polynomial form originally given by Freedman . The superspace description given here corresponds to the component formulation of Stelle and West.

We now consider the coupling of supergravity to spinor-scalar-pseudoscalar matter. This supermultiplet is described by a chiral superfield ϕ

$$\mathcal{D}_{\dot\alpha}\,\phi = 0 \qquad , \quad \mathcal{D}_{\alpha}\,\phi^* = 0 \tag{5.11}$$

As action, to add to that of supergravity, one can take

$$\int E\,\phi\,\phi^* + E\left(\frac{1}{R}F(\phi) + \frac{1}{R^*}F(\phi^*)\right) \tag{5.12}$$

where $F(\phi)$, the potential, is a (real) function of ϕ . In order to derive the equations of motion it is simplest to satisfy the conditions (5.11) by setting, according to (3.28),

$$\begin{aligned} \phi &= \left(\mathcal{D}_{\dot\alpha}\mathcal{D}^{\dot\alpha} - 8R\right)\mathcal{U} \\ \phi^* &= \left(\mathcal{D}^{\alpha}\mathcal{D}_{\alpha} - 8R^*\right)\mathcal{U}^* \end{aligned} \tag{5.13}$$

The superfields U and U^* can be subjected to arbitrary variations. One finds the equation of motion

$$\left(\mathcal{D}^\alpha \mathcal{D}_\alpha - 8\mathcal{R}^*\right)\phi - 8\,F'(\phi^*) = 0 \tag{5.14}$$

and its complex conjugate. It is not difficult to verify that (5.11) and (5.14) imply that the supercurrent

$$\mathcal{J}_{\alpha\dot\alpha} = i\,\phi\left(\overrightarrow{\mathcal{D}}_{\alpha\dot\alpha} - \overleftarrow{\mathcal{D}}_{\alpha\dot\alpha}\right)\phi^* + \frac{1}{2}\mathcal{D}_\alpha\phi\cdot\mathcal{D}_{\dot\alpha}\phi^* + 2G_{\alpha\dot\alpha}\,\phi\phi^* \tag{5.15}$$

satisfies

$$\mathcal{D}^\alpha \mathcal{J}_{\alpha\dot\alpha} = \mathcal{D}_{\dot\alpha}\, S^* \tag{5.16}$$

where

$$S = 6\,F(\phi) - 2\,\phi\,F'(\phi)\,, \qquad \mathcal{D}_{\dot\alpha} S = 0 \tag{5.17}$$

The matter superfields $J_{\alpha\dot\alpha}$ and S, S^* (which contain the energy momentum tensor and the spinor current satisfy an identity analogous to that (see (3.18)) satisfied by the supergravity superfields $G_{\alpha\dot\alpha}$ and R, R^* (which contain the Einstein tensor and the Rarita-Schwinger operator). Indeed the supergravity equations state the proportionality of these two sets of fields

$$G_{\alpha\dot\alpha} = c\cdot \mathcal{J}_{\alpha\dot\alpha}\,, \qquad \mathcal{R} = c\,S \tag{5.18}$$

If one attibutes to the superfield ϕ the conformal property

$$\delta\phi = -2\Sigma\,\phi \tag{5.19}$$

one finds that the first (kinetic) term in the action (5.12) is conformally invariant. For the potential term, one obtains

$$\delta\int\frac{E}{\mathcal{R}}F(\phi) = 2\int\frac{E}{\mathcal{R}}\cdot S\cdot\Sigma \tag{5.20}$$

The superfield S, given by (5.17), appears as a measure of the violation of conformal invariance in the matter action. For $F(\phi) \propto \phi^3$, one has S = 0 and the entire matter action is conformally invariant. In general, however, for instance if the matter supermultiplet has a mass ($F \propto \phi^2$)R will not vanish. The coupling described by (5.12) is only one of many possible forms which have been studied in detail in terms of component fields (see the lectures by S. Ferrara and P. Van Nieuwenhuizen, where many references can be found). It is an amusing exercise to translate the other forms into superspace language.

CHIRAL GAUGE

We now indicate briefly the procedure by which the constraints can actually be solved. We give only the starting point of this procedure and refer to for further details. It is convenient to choose a gauge in which certain components of the inverse vielbein have a simple form

$$E_{\dot\alpha}{}^{M} = \rho\, \delta_{\dot\alpha}{}^{M} \tag{6.24}$$

where ρ is a function of x, θ and $\bar\theta$. Then, for a scalar superfield U

$$\mathcal{D}^{\dot\alpha} \mathcal{U} = \rho\, \partial^{\dot\alpha} \mathcal{U} \tag{6.25}$$

where $\partial^{\dot\alpha}$ is an ordinary anticommuting derivative. We see that in this gauge a chiral superfield is actually independent of $\bar\theta$: we call it the chiral gauge.

Let us now take the torsion constraint

$$T_{\dot\alpha\dot\beta}{}^{N} = 0 \tag{6.26}$$

or, more explicit ly

$$E_{\dot\alpha}{}^{M}\partial_M E_{\dot\beta}{}^{N} + E_{\dot\beta}{}^{M}\partial_M E_{\dot\alpha}{}^{N} + \left(\phi_{\dot\alpha\dot\beta}{}^{\dot\gamma} + \phi_{\dot\beta\dot\alpha}{}^{\dot\gamma}\right) E_{\dot\gamma}{}^{N} = 0 \tag{6.27}$$

It can be solved for the connection, with the result

$$\phi_{\dot\alpha\dot\beta\dot\gamma} = \partial_{\dot\beta}\rho\, \varepsilon_{\dot\alpha\dot\gamma} + \partial_{\dot\gamma}\rho\, \varepsilon_{\dot\alpha\dot\beta} \tag{6.28}$$

This component of the connection enters in the expression for $R_{\dot\varepsilon\dot\alpha\dot\beta\dot\gamma}$. Making use of the complex conjugate of the first of (3.17), this is expressible in terms of the chiral superfield R. In this way one finds

$$\partial_{\dot\alpha}\partial^{\dot\alpha}\rho^2 = -8R \tag{6.29}$$

which is obviously chiral in the gauge (6.25). This equation shows that it would not have been possible, in general, to choose the gauge so that in (6.24) ρ was equal to unity. We have now all the ingredients to calculate an expression of the form (3.28). Using (6.24), (6.25), (6.28) and (6.29), the various terms quadratic in ρ and its derivatives combine in the very simple expression

$$\left(\mathcal{D}_{\dot\gamma}\mathcal{D}^{\dot\gamma} - 8R\right)\mathcal{U} = \partial_{\dot\gamma}\partial^{\dot\gamma}(\rho^2\mathcal{U}) \tag{6.30}$$

Again, the result is obviously chiral, as we know it must be. In the chiral gauge an integral of the form (4.13) takes a particularly simple form. Remembering that a θ integration can be written as a θ derivative, we have

$$-\frac{1}{8}\int dx\, d\theta\, d\bar{\theta}\, \frac{E}{R}\,\phi = -\frac{1}{8}\int dx\, d\theta\, \partial_{\dot\alpha}\partial^{\dot\alpha}\left(\frac{E}{R}\phi\right) \cdot$$
$$= -\frac{1}{8}\int dx\, d\theta \left(\partial_{\dot\alpha}\partial^{\dot\alpha}E\right)\frac{1}{R}\phi \equiv \int dx\, d\theta\, \mathcal{E}\,\phi \tag{6.31}$$

where we have used the chirality conditions

$$\partial_{\dot\alpha}\phi = \partial_{\dot\alpha} R = 0 \tag{6.32}$$

We see that the density $\mathcal{E}$ appropriate to a chiral superfield satisfies

$$\partial_{\dot\alpha}\partial^{\dot\alpha}E = -8\,R\,\mathcal{E} \tag{6.33}$$

$$\mathcal{E} = \partial_{\dot\alpha}\partial^{\dot\alpha}E \Big/ \partial_{\dot\alpha}\partial^{\dot\alpha}\rho^2 \tag{6.34}$$

These relations can indeed be verified explicitly in terms of components.

The chiral gauge can be obtained from a general gauge by making a suitable Lorentz transformation accompanied by a general coordinate transformation. One could have chosen a different chiral gauge obtained from the above by complex conjugation, in which the vielbein with an undotted index had a simple form analogous to (6.24). These two chiral gauges are the analogues in curved superspace, of the two representations in flat superspace in which the two types of chiral fields are respectively independent of θ and of $\bar\theta$. In flat superspace these two representations are related by a pure imaginary translation. In curved superspace the two chiral gauges are related by a pure imaginary general coordinate transformation. Two quantities, like $E_\alpha{}^M$ and $E_{\dot\alpha}{}^M$, which ordinarily are related by complex conjugation, in the gauge (6.24) are obtained from each other by performing a complex conjugation followed by a pure imaginary general coordinate transformation. Thus, when (6.25) is valid, one has

$$E_\alpha{}^M\partial_M = e^{-iU^M\partial_M}\,\rho^*\,\partial_\alpha\, e^{iU^M\partial_M} \tag{6.35}$$

where $-iU^M(x,\theta)$ is the pure imaginary parameter of the general coordinate transformation relating the two gauges. The two equations (6.24) and (6.35) express all components of the inverse supervielbein with spinorial tangent space indices in terms of the two superfields ρ and U^M. Using these expressions in the supertorsion constraints one can express the other components of the supervielbein (and therefore also all components of the super connection) in terms of the same two superfields. Furthermore, it is easy to see that the conditions (6.24) determining the chiral gauge does not fix the gauge completely. Using the remaining gauge arbitrariness one can impose conditions such that everything is expressed in terms of the components U^m alone (vectorial values of M only). At this point the constraints are completely solved in terms of this superfield U^m and the action of supergravity can be expressed in terms of it.

DIVERGENCES IN SUGRA

We will consider possible counterterms at the one, two, and three loop levels. These are simplified by the mass shell conditions,

$$G_{\alpha\dot\beta} = 0 \qquad R = 0$$

$$D^\alpha W_{\alpha\beta\gamma} = 0 \qquad \bar D_{\dot\delta} W_{\alpha\beta\gamma} = 0$$

Thus, counterterms can only be constructed from W, W^*, D, and $\bar D$. These must be multiplied by the correct density; ie. $E\, d^8z$ with dimension -6 for a general scalar, $\frac{E}{R} d^8z$ with dimension -5 for a chiral scalar. W and D have dimensions $5/2$ and $1/2$ respectively, and the entire term must come out with dimension $2L-2$ as before (L = # of loops).

<u>One loop</u> Here we have only one possible term,

$$\int W_{\alpha\beta\gamma} W^{\alpha\beta\gamma} \frac{E}{R} d^8z$$

As in ordinary gravity, this can be shown

to vanish by an analogous "SUSY Gauss-Bonnet" formula:

$$\int \left[\frac{E}{R} W_{\alpha\beta\gamma} W^{\alpha\beta\gamma} - 4E\left(G_{\alpha\dot\beta} G^{\alpha\dot\beta} - 4 R^* R \right) \right] d^8 z = 0$$

<u>Two loops</u> Here again there is only one possible term with correct dimension that does not vanish immediately due to the equations of motion,

$$\int (D^\alpha W_{\beta\gamma\delta}) W_\alpha{}^\beta{}_\epsilon \, W_{\gamma\delta}{}^\epsilon \; E \, d^8 z$$

however, integrating by parts gives

$$= -\int W_{\beta\gamma\delta} \, W_\alpha{}^\beta{}_\epsilon \, (D^\alpha W_{\gamma\delta}{}^\epsilon) \, E \, d^8 z$$

$$= -\int W^\epsilon{}_{\gamma\delta} \, W_{\alpha\epsilon}{}^\beta \, (D^\alpha W_{\gamma\delta\beta}) E d^8 z = 0$$

(negative of above)

<u>Three loops</u> Now there is a possible non vanishing term,

$$\int W_{\alpha\beta\gamma} W^{\alpha\beta\gamma} W^*_{\dot\alpha\dot\beta\dot\gamma} W^{*\dot\alpha\dot\beta\dot\gamma} \; E \, d^8 z$$

A component of this contains $C \, C \, C^* \, C^*$ which is the square of the Bel-Robinson tensor

Physics 250 Notes
Bruno Zumino

INVERSE VIELBEIN FORMULAS

For purposes of dimensional reduction it is convenient to express the Einstein action in a way that emphasizes the <u>inverse</u> viel-bein. First we need some identities involving the torsion.

$$T_{mn}{}^{a} = \partial_m e_n{}^a - \partial_n e_m{}^a + e_n{}^b \omega_{mb}{}^a - e_m{}^b \omega_{nb}{}^a$$

$$T_{fg}{}^{a} = e_f{}^m e_g{}^n (\partial_m e_n{}^a - \partial_n e_m{}^a) + \omega_{fg}{}^a - \omega_{gf}{}^a$$

Now $\partial_m e_n{}^a = -(\partial_m e_b{}^\ell) e_\ell{}^a e_n{}^b$

inverse vielbein $\quad (\partial_f \equiv e_f{}^m \partial_m)$

so $T_{fg}{}^a = (-\partial_f e_g{}^\ell + \partial_g e_f{}^\ell) e_\ell{}^a + \omega_{fg}{}^a - \omega_{gf}{}^a$

or: $$\boxed{T_{ab}{}^m = -\partial_a e_b{}^m + \partial_b e_a{}^m + \omega_{ab}{}^m - \omega_{ba}{}^m} \quad (1)$$

(compare with first line)

Taking the trace, we find :

$$T_{ab}{}^{b} = \partial_m e_a{}^m - e_m{}^b \partial_a e_b{}^m - \omega_{ba}{}^b$$

(since $\omega_{ab}{}^b = 0$)

$$\boxed{T_{ab}{}^{b} = \frac{1}{e}\partial_m (e\, e_a{}^m) - \omega_{ba}{}^b} \quad (2)$$

Writing the curvature tensor in terms of quantities having only tangent space indicies also introduces the inverse vielbein:

$$R_{mna}{}^b = \partial_m \omega_{na}{}^b - \partial_n \omega_{ma}{}^b - \omega_{ma}{}^c \omega_{nc}{}^b + \omega_{na}{}^c \omega_{mc}{}^b$$

$$R_{fga}{}^b = \underbrace{e_f{}^m e_g{}^n (\partial_m \omega_{na}{}^b - \partial_n \omega_{ma}{}^b)} - \omega_{fa}{}^c \omega_{gc}{}^b + \omega_{ga}{}^c \omega_{fc}{}^b$$

$$e_f{}^m (e_g{}^n \partial_m \omega_{na}{}^b) = \partial_f \omega_{ga}{}^b - (\partial_f e_g{}^n) \omega_{na}{}^b$$

$$\partial_f \omega_{ga}{}^b - \partial_g \omega_{fa}{}^b \underbrace{- (\partial_f e_g{}^n - \partial_g e_f{}^n)} \omega_{na}{}^b$$

$$T_{fg}{}^n - \omega_{fg}{}^n + \omega_{gf}{}^n \quad \text{from (1)}$$

so,

$$\boxed{\begin{aligned} R_{cda}{}^b &= \partial_c \omega_{da}{}^b - \partial_d \omega_{ca}{}^b + T_{cd}{}^e \omega_{ea}{}^b \\ &\quad - (\omega_{cd}{}^e - \omega_{dc}{}^e)\omega_{ea}{}^b - \omega_{ca}{}^e \omega_{de}{}^b + \omega_{da}{}^e \omega_{ce}{}^b \end{aligned}}$$

Fully contracting this gives the curvature scalar:

$$R = \partial_a \omega_b{}^{ab} - \partial_b \omega_a{}^{ab} + T_{ab}{}^{e} \omega_e{}^{ab} - \omega_{ab}{}^{e} \omega_e{}^{ab}$$
$$+ \omega_{ba}{}^{e} \omega_e{}^{ab} - \omega_a{}^{ae} \omega_{be}{}^{b} + \omega_{ba}{}^{e} \omega^{a}{}_{e}{}^{b}$$

$$= 2 \partial_a \omega_b{}^{ab} + T_{abc} \omega^{cab} - 2 \omega_{abc} \omega^{cab}$$
$$+ \omega_{abc} \omega^{cab} + \omega_a{}^{ae} \omega_b{}^{b}{}_{c}$$

$$= 2 \partial_a \omega_b{}^{ab} + T_{abc} \omega^{cab} - \omega_{abc} \omega^{cab}$$
$$+ \omega_a{}^{ac} \omega_b{}^{b}{}_{c}$$

Now do an integration by parts on the first term:

$$e(e_a{}^m \partial_m \omega_b{}^{ab}) \overset{p.i.}{=} - \partial_m (e e_a{}^m) \omega_b{}^{ab}$$
$$= -(T_{ab}{}^{b} \omega_c{}^{ac} + \omega_{ba}{}^{b} \omega_c{}^{ac}) e$$
by (2)

If the torsion vanishes, we have:

$$eR \overset{p.i.}{=} (-2 \omega_{ba}{}^{b} \omega_c{}^{ac} - \omega_{abc} \omega^{cab} + \omega_a{}^{ac} \omega_b{}^{b}{}_{c}) e$$

$$\boxed{eR \overset{p.i.}{=} e(-\omega_a{}^{ac} \omega_b{}^{b}{}_{c} - \omega_{abc} \omega^{cab})} \quad (3)$$

KALUZA-KLEIN MODEL

This is an illustrative example of dimensional reduction. One begins with Einstein's gravity in five dimensions and makes the assumption that the fields are independent of the fifth dimension. What results is a theory of gravity in four dimensions coupled to additional fields.

fünfbein : $E_M{}^A$

$$M, A = 1,2,3,4,5 \quad ; \quad m, a = 1,2,3,4$$

We can make a local Lorentz transformation on the tangent space vector $E_5{}^A$ so that

$$E_5{}^A = (0, 0, 0, 0, E_5{}^5)$$

This corresponds to a choice of gauge where $E_5{}^a = 0$. Consider the effect of general coordinate transformations on the remaining parts of the fünfbein :

$$\delta E_M{}^A = \xi^L \partial_L E_M{}^A + \partial_M \xi^L E_L{}^A$$

Since $\partial_5 \cdot = 0$, we have,

$$\delta E_m{}^a = \xi^\ell \partial_\ell E_m{}^a + \partial_m \xi^\ell E_\ell{}^a$$

(since $E_5{}^a = 0$)

$$\delta E_m{}^5 = \xi^\ell \partial_\ell E_m{}^5 + \partial_m \xi^\ell E_\ell{}^5 + \partial_m \xi^5 E_5{}^5$$

$$\delta E_5{}^5 = \xi^\ell \partial_\ell E_5{}^5$$

The first has the correct transformation law of the usual vierbein; the third, that of a four dimensional scalar. Define:

$$B\, e_m{}^a = E_m{}^a$$

$$A = E_5{}^5$$

Later we'll see that A and B must be related in a particular way. Finally, if we define,

$$v_m = \frac{E_m{}^5}{E_5{}^5}$$

then,

$$\delta v_m = \xi^\ell \partial_\ell v_m + \partial_m \xi^\ell v_\ell + \partial_m \Lambda$$

where $\Lambda = \xi^5$ takes on the role of the parameter of a <u>gauge</u> transformation of the 4-vector field v_m.

$$E_M{}^A = \begin{pmatrix} B e_m{}^a & A v_m \\ 0 & A \end{pmatrix} \qquad E_5{}^a = 0$$

by multiplication we verify the inverse is,

$$E_A{}^M = \begin{pmatrix} \frac{1}{B} e_a{}^m & -\frac{1}{B} v_a \\ 0 & \frac{1}{A} \end{pmatrix} \qquad e_a{}^m e_m{}^b = \delta_a{}^b$$

$$E_5{}^m = 0$$

$$\det E_M{}^A = A B^4 \det e_m{}^a$$

Now comes the tedious part of computing the connection.

$$C_{BCA} = (\partial_M E_{NA} - \partial_N E_{MA}) E_B{}^M E_C{}^N = -C_{CBA}$$

$$C_{bca} = (\partial_m E_{na} - \partial_n E_{ma}) E_b{}^m E_c{}^n$$

$$= \frac{1}{B} C_{bca}(e) + \frac{\partial_m B}{B^2} (\eta_{ac} e_b{}^m - \eta_{ab} e_c{}^m)$$

$$C_{bca}(e) = (\partial_m e_{na} - \partial_n e_{ma}) e_b{}^m e_c{}^n$$

$$C_{bc5} = (\partial_m E_{n5}) E_b{}^m E_c{}^n + (\partial_m E_{55}) E_b{}^m E_c{}^5$$

$$- (c \leftrightarrow b)$$

$$C_{bc5} = \frac{1}{B^2}(\partial_m A v_n)e_b{}^m e_c{}^n - \frac{1}{B^2}(\partial_m A)e_b{}^m v_c$$

$$= \frac{A}{B^2} v_{mn} e_b{}^m e_c{}^n \qquad - (b \leftrightarrow c)$$

$$(v_{mn} = \partial_m v_n - \partial_n v_m)$$

$$C_{b5a} = (\partial_m E_{\underset{\substack{\downarrow \\ n}}{Na}}) E_b{}^m \underbrace{E_5{}^{\underset{\substack{\downarrow \\ n}}{N}}}_{0} - (\partial_n E_{Ma}) E_b{}^M \underbrace{E_5{}^n}_{0}$$

$$= 0$$

$$C_{b55} = (\partial_m E_{\underset{\substack{\downarrow \\ 5}}{N5}}) E_b{}^m E_5{}^{\underset{\substack{\downarrow \\ 5}}{N}} - (\partial_n E_{M5}) E_b{}^M \underbrace{E_5{}^n}_{0}$$

$$= (\partial_m A) \frac{e_b{}^m}{BA}$$

$$C_{555} = 0$$

$$\Omega_{BCA} = -\frac{1}{2}(C_{BCA} - C_{CAB} + C_{ABC})$$

$$\Omega_{bca} = \frac{1}{B}\omega_{bca} - \frac{1}{2}\frac{1}{B^2}(\cancel{\eta_{ac}\partial_b} - \eta_{ab}\partial_c -$$

$$- \eta_{ba}\partial_c + \eta_{bc}\partial_a + \eta_{cb}\partial_a - \cancel{\eta_{ca}\partial_b})B$$

$$= \frac{1}{B}\omega_{bca} + \frac{1}{B^2}(\eta_{ab}\partial_c - \eta_{bc}\partial_a)B$$

$$\Omega_{b5a} = -\tfrac{1}{2} C_{ab5} = -\tfrac{1}{2}\frac{A}{B^2} v_{mn} e_a{}^m e_b{}^n$$

$$\Omega_{5ca} = \tfrac{1}{2} C_{ca5} = \tfrac{1}{2}\frac{A}{B^2} v_{mn} e_c{}^m e_a{}^n$$

$$\Omega_{55a} = C_{5a5} = -(\partial_m A)\frac{e_a{}^m}{BA}$$

When we use (3) to write the action we get,

$$\int -\tfrac{1}{2} E R \, d^5x = \int -\tfrac{1}{2} E\left(-\Omega_{ABC}\Omega^{CAB} - \Omega_A{}^{AC}\Omega_B{}^B{}_C\right)$$

$$= \left(\int dx^5\right)\int -\tfrac{1}{2} A B^4 e \left\{ -\frac{1}{B^2}\omega_{abc}\omega^{cab} - \frac{1}{B^2}\omega_a{}^{ac}\omega_b{}^b{}_c + \cdots \right\} d^4x$$

Dropping the factor $\int dx^5$, we get the correct 4 dimensional gravity action if we demand,

$$A B^4 \frac{1}{B^2} = 1$$

From now on we will replace B by $A^{-1/2}$.

$$\Omega_A{}^{A5} = 0 \quad , \quad \Omega_A{}^{Ac} = \Omega_a{}^{ac} + \Omega_5{}^{5c}$$

$$\Omega_A{}^{Ac} = \frac{1}{B}\,\omega_a{}^{ac} - \frac{3}{B^2}(\partial^c B) - \frac{1}{BA}(\partial^c A)$$

$$= A^{1/2}\left\{\omega_a{}^{ac} + \frac{1}{2}(\partial^c A)\frac{1}{A}\right\}$$

$$\Omega_A{}^{Ac}\,\Omega_B{}^{B}{}_c = A\left\{\omega_a{}^{ac}\,\omega_b{}^{b}{}_c + \omega_a{}^{ac}(\partial_c A)\frac{1}{A} + \frac{1}{4}(\partial_c A)^2\frac{1}{A^2}\right\}$$

$$\Omega_{ABC}\,\Omega^{CAB} = \Omega_{abc}\,\Omega^{cab} + \Omega_{ab5}\,\Omega^{5ab}$$

$$+ \Omega_{a5c}\,\Omega^{ca5} + \Omega_{5bc}\,\Omega^{c5b} + \Omega_{55c}\underbrace{\Omega^{c55}}_{0}$$

$$+ \Omega_{5b5}\,\Omega^{55b} + \underbrace{\Omega_{a55}}_{0}\,\Omega^{5a5}$$

$$\Omega_{abc}\,\Omega^{cab} = \frac{1}{B^2}\,\omega_{abc}\,\omega^{cab} + \frac{1}{B^3}\,\omega_{abc}\left(\cancel{\eta^{cb}\partial^a} - \eta^{ca}\partial^b\right)B$$

$$+ \frac{1}{B^3}\left(\eta_{ac}\partial_b - \cancel{\eta_{ab}\partial_c}\right)B\,\omega^{cab} +$$

$$\frac{1}{B^4}\left(\eta_{ac}\partial_b - \eta_{ab}\partial_c\right)B\left(\eta^{cb}\partial^a - \eta^{ca}\partial^b\right)B$$

-1 $\quad +1 \quad +1 \quad -4$

$$= -3(\partial_a B)^2$$

$$\Omega_{abc}\Omega^{cab} = A\left\{\omega_{abc}\omega^{cab} + \frac{2}{AB^3}\omega_a{}^{ab}\partial_b B - \frac{3}{AB^4}(\partial_a B)^2\right\}$$

$$= A\left\{\omega_{abc}\omega^{cab} - \frac{1}{A}\omega_a{}^{ab}\partial_b A - \frac{3}{4}\frac{1}{A^2}(\partial_a A)^2\right\}$$

$$\Omega_{ab5}\Omega^{5ab} = \left(-\frac{1}{2}\frac{A}{B^2}\mathcal{V}_{ab}\right)\left(\frac{1}{2}\frac{A}{B^2}\mathcal{V}^{ab}\right) = -\frac{1}{4}A^4(\mathcal{V}_{ab})^2$$

$$= \Omega_{5bc}\Omega^{c5b} = -\Omega_{a5c}\Omega^{ca5}$$

$$\Omega_{565}\Omega^{55b} = -\Omega_{55b}\Omega^{55b} = -\frac{1}{(BA)^2}(\partial_a A)^2$$

$$= -\frac{1}{A}(\partial_a A)^2$$

$$\Omega_{ABC}\Omega^{CAB} = A\left\{\omega_{abc}\omega^{cab} - \frac{1}{A}\omega_a{}^{ab}(\partial_b A) - \frac{7}{4}\frac{1}{A^2}(\partial_a A)^2 - \frac{1}{4}A^3(\mathcal{V}_{ab})^2\right\}$$

$$\Omega_{ABC}\Omega^{CAB} + \Omega_A{}^{AC}\Omega_B{}^B{}_C = A\left\{\omega_{abc}\omega^{cab} + \omega_a{}^{ac}\omega_b{}^b{}_c - \frac{3}{2}\frac{1}{A^2}(\partial_a A)^2 - \frac{1}{4}A^3(\mathcal{V}_{ab})^2\right\}$$

The four dimensional action becomes:

$$\int d^4x \left\{ -\tfrac{1}{2} eR - \tfrac{3}{4} \tfrac{1}{A^2} (\partial_a A)^2 e - \tfrac{1}{8} A^3 (v_{ab})^2 e \right\}$$

From the fünfbein,

$$E_M{}^A = \begin{pmatrix} A^{-1/2} e_m{}^a & A v_m \\ 0 & A \end{pmatrix}$$

we see that (in perturbation theory) A must have the form $1 + \cdots$. Therefore, it would have been more natural to use the field ϕ where,

$$A = e^{\sqrt{\frac{2}{3}}\phi}$$

Also, if we redefine v_{mn} by a factor of $\sqrt{2}$ then

$$\int d^5x \left(-\tfrac{1}{2} ER\right) \longrightarrow \int d^4x\, e \left\{ -\tfrac{1}{2} R - \tfrac{1}{2} (\partial_a \phi)^2 \right.$$

$$\left. \underbrace{-\tfrac{1}{4} e^{\sqrt{6}\phi}}_{1 + \sqrt{6}\phi + \cdots} (F_{ab})^2 \right\}$$

SPINORS IN MANY DIMENSIONS

For purposes of dimensional reduction we'll need to know general properties of representations of the Clifford algebra

$$\{\Gamma_A, \Gamma_B\} = 2\eta_{AB}$$

$$\eta_{AB} = (\underbrace{+\cdots+}_{p}\ \underbrace{-\cdots-}_{q}) \qquad p+q = D$$

We only need to consider the case $D=$ even, since, in $D+1$ dimensions we define

$$\Gamma_{D+1} = \Gamma_1 \cdots \Gamma_D$$

from which follows,

$$\{\Gamma_{D+1}, \Gamma_A\} = 0 \qquad \Gamma_{D+1}^2 = \pm 1$$

$$A = 1, \ldots, D$$

REAL REPRESENTATIONS

There is always a hermitean $2^{D/2}$ dim. repr. in the case $p=D$ $q=0$. By multiplying q of them by i we get a repr. of the case (p,q) with p hermitean and

(cont.)

q antihermitean. If these matrices are also <u>real</u>, then p are symmetric and q are antisymmetric. In this case, we can use the fact that the set of real matrices,

$$1 \,, \quad \Gamma_A \,, \quad \underset{A<B}{\Gamma_A \Gamma_B} \,, \quad \ldots \,, \quad \Gamma_1 \Gamma_2 \cdots \Gamma_D$$

span the space of all real $2^{D/2} \times 2^{D/2}$ matrices. Each of these is either symmetric or antisymmetric. But we know there are exactly $\frac{1}{2}\, 2^{D/2}(2^{D/2}-1)$ antisymmetric $2^{D/2} \times 2^{D/2}$ matrices. Counting the number of antisymmetric matrices in the above list will give us conditions on (p, q) such that real reps. exist.

Example : $D=4$ # antisymmetric $= 6$

$p=4 \quad q=0 \qquad 1 \,, \; \Gamma_A \,, \; \Gamma_A\Gamma_B \,, \; \Gamma_A\Gamma_B\Gamma_C \,, \; \Gamma_A\Gamma_B\Gamma_C\Gamma_D$

$$\begin{array}{ccccc} S & S & A & A & S \\ \binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & \binom{4}{4} \\ & & \overset{\shortparallel}{6} & \overset{\shortparallel}{4} & \end{array}$$

$\left(\begin{array}{l} S = \text{sym.} \\ A = \text{antisym.} \end{array}\right)$

$$6 + 4 = 10 \neq 6 \,!$$

So a real rep. of

$$\{\Gamma_A, \Gamma_B\} = 2\delta_{AB} \qquad D=4$$

does not exist.

$p = 3 \quad q = 1 \qquad \gamma_0^T = -\gamma_0 \quad \gamma_i^T = \gamma_i$

	1	γ_0	$\gamma_0\gamma_i$	$\gamma_0\gamma_i\gamma_j$	$\gamma_0\gamma_1\gamma_2\gamma_3$		
		γ_i	$\gamma_i\gamma_j$	$\gamma_1\gamma_2\gamma_3$			
S	1	3	3	3	0	=	10
A	0	1	3	1	1	=	6 ✓

A useful fact to know is that the parity of the permutations 21, 321, 4321, etc. have the sequence $-, -, +, +, -, -, ++, \ldots$ (as can be seen in the first example). This parity is $(-1)^{\frac{k(k-1)}{2}}$ where k = # of reversed objects as can be easily seen by counting the number of even cycles necessary to restore the permutation to the identity.

The second example shows that in general, the matrices in each category

$$\Gamma^{(k)} = \{ \Gamma_{A_1} \Gamma_{A_2} \cdots \Gamma_{A_k} \}$$

do <u>not</u> have the same symmetry. We can rearrange the matrices into categories that <u>do</u> have a definite symmetry by multiplying all of them by τ, where,

(cont.)

$$\tau = \Gamma_{p+1} \cdots \Gamma_{p+q}$$

$$\tau^{-1} = (-1)^q \Gamma_{p+q} \cdots \Gamma_{p+1} = \tau^T$$

$$= (-1)^q (-1)^{\frac{1}{2}q(q-1)} \Gamma_{p+1} \cdots \Gamma_{p+q}$$

$$= (-1)^{\frac{1}{2}q(q+1)} \tau$$

Now $\quad \tau \Gamma_A \tau^{-1} = \Gamma_{p+1} \cdots \Gamma_{p+q} \Gamma_A \Gamma_{p+q} \cdots \Gamma_{p+1} (-1)^q$

$$\begin{cases} = (-1)^q \Gamma_A = (-1)^q \Gamma_A^T & A = 1, \dots, p \\ = -(-1)^q \Gamma_A = (-1)^q \Gamma_A^T & A = p+1, \dots, p+q \end{cases}$$

So $\quad \tau \Gamma_A \tau^{-1} = (-1)^q \Gamma_A^T$

(τ is the analogue of γ_0)

Moreover, $\quad \tau \, 1 \, \tau^{-1} = 1^T$

$$\tau \Gamma_A \tau^{-1} = (-1)^q \Gamma_A^T$$

$$\tau \Gamma_A \Gamma_B \tau^{-1} = \tau \Gamma_A \tau^{-1} \tau \Gamma_B \tau^{-1}$$

$$= (-1)^{2q} \Gamma_A^T \Gamma_B^T = -(-1)^{2q} (\Gamma_A \Gamma_B)^T$$

$$\tau \Gamma_A \Gamma_B \Gamma_C \tau^{-1} = (-1)^{3q} (\Gamma_C \Gamma_B \Gamma_A)^T$$

$$= -(-1)^{3q} (\Gamma_A \Gamma_B \Gamma_C)^T$$

$$\vdots$$

$$\tau \Gamma^{(k)} \tau^{-1} = (-1)^{\frac{1}{2}k(k-1) + kq} \Gamma^{(k)T}$$

using the parity rule given before.

$$\tau\,\Gamma^{(k)} = (-1)^{\frac{1}{2}k(k-1)+kq}\,\Gamma^{(k)T}\,\tau$$

$$(-kq) \qquad \left((-1)^{\frac{1}{2}q(q+1)}\,\tau^T\right)$$

$$\tau\,\Gamma^{(k)} = (-1)^{\frac{1}{2}\underbrace{[k^2-k-2kq+q^2+q]}}\left(\tau\,\Gamma^{(k)}\right)^T$$

$$(k-q)^2-(k-q)$$

$$= (k-q)(k-q-1)$$

so $$\tau\,\Gamma^{(k)} = (-1)^{\frac{1}{2}(k-q)(k-q-1)}\left(\tau\,\Gamma^{(k)}\right)^T$$

To count the number of antisymmetric matrices in the set,

$$\tau,\ \tau\Gamma_A,\ \tau\Gamma^{(2)},\ \ldots,\ \tau\,\Gamma^{(p+q)}$$

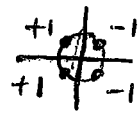
we note that

$$\epsilon_k = \frac{1}{2}-\frac{1}{2}(-1)^{\frac{1}{2}(k-q)(k-q-1)} = \begin{cases} 0 & \text{if sym.} \\ 1 & \text{if antisym.} \end{cases}$$

and since the category $\tau\,\Gamma^{(k)}$ contains $\binom{D}{k}$ matrices, we have (N = # antisymmetric matrices)

$$N = \sum_{k=0}^{D} e_k \binom{D}{k} = \frac{1}{2} \sum_{k=0}^{D} \left[1 - (-1)^{\frac{1}{2}(k-q)(k-q-1)} \right] \binom{D}{k}$$

$$(-1)^{\frac{1}{2}(k-q)(k-q-1)} = -\sqrt{2} \cos \frac{\pi}{4}(2k-2q+3)$$



$$= -\frac{1}{\sqrt{2}} \left[e^{i\frac{3}{4}\pi} \left(e^{i\frac{\pi}{2}}\right)^{k-q} + \text{c.c.} \right]$$

$$= \frac{1}{2} \left[(1-i)\, i^{k-q} + (1+i)(-i)^{k-q} \right]$$

$$N = \frac{1}{2} 2^D - \left\{ \frac{1}{4}(1-i)(i^{-q}) \underbrace{\sum_k i^k \binom{D}{k}}_{(1+i)^D} + \text{c.c.} \right\}$$

$$= \frac{1}{2} 2^D + \left\{ \frac{1}{4}\sqrt{2}\, e^{i\frac{3}{4}\pi}\, e^{i\frac{\pi}{2}(-q)} (\sqrt{2})^D e^{i\frac{\pi}{4}D} + \text{c.c.} \right\}$$

$$= 2^{D-1} + 2^{\frac{D+1}{2}-1} \cos\left[\frac{\pi}{4}(3 \underbrace{-2q + D}_{p-q}) \right]$$

$$N = 2^{D-1} + 2^{\frac{D-1}{2}} \cos \frac{\pi}{4}(p-q+3)$$

Since the other way of counting N gives

$$\frac{1}{2} 2^{D/2}(2^{D/2}-1) = 2^{D-1} - 2^{\frac{D-1}{2}} \frac{1}{\sqrt{2}}$$

we need to have $\quad \sqrt{2} \cos \frac{\pi}{4}(p-q+3) = -1$

or $\quad p-q = 0$ or $2 \pmod 8$

This shows that for $q=1$, real reps. exist only for $D = 2, 4, 10, 12, \ldots$

CONSTRUCTION OF REAL REPS.

We will let $C(p,q)$ denote the hermitean rep. of the Clifford algebra

$$\{\Gamma_A, \Gamma_B\} = 2\,\delta_{AB}$$

having p real symmetric matrices P_i, and q imaginary antisymmetric matrices Q_j. Larger reps. will be constructed as direct products of these with the usual Pauli matrices. For $D=2$ there are two possibilities:

$C(2,0) \qquad P_1 = \sigma_1 \quad P_2 = \sigma_3$

$C(1,1) \qquad P_1 = \sigma_1 \quad Q_1 = \sigma_2$

From $C(p,q)$ we construct $C(q+2,p)$ by:

$$P_j = Q_j \times \sigma_2 \qquad\qquad Q_i = P_i \times \sigma_2$$

$$P_{q+1} = 1 \times \sigma_1$$

$$P_{q+2} = 1 \times \sigma_3 \qquad \begin{pmatrix} i = 1, \ldots, p \\ j = 1, \ldots, q \end{pmatrix}$$

Similarily, we can construct $c(p+1, q+1)$:

$$P_i = P_i \times \sigma_3 \qquad Q_j = Q_j \times \sigma_3$$
$$P_{p+1} = 1 \times \sigma_1 \qquad Q_{q+1} = 1 \times \sigma_2$$

These constructions give us the two kinds of real reps. in four dimensions:

$$(2,0) \longrightarrow (2+1, 0+1) = (3,1)$$

$$P_1 = \sigma_1 \times \sigma_3$$
$$P_2 = \sigma_3 \times \sigma_3 \qquad Q_1 = 1 \times \sigma_2$$
$$P_3 = 1 \times \sigma_1$$

$$(2,0) \longrightarrow (0+2, 2) = (2,2)$$

$$P_1 = 1 \times \sigma_1 \qquad Q_1 = \sigma_1 \times \sigma_2$$
$$P_2 = 1 \times \sigma_3 \qquad Q_2 = \sigma_3 \times \sigma_2$$

No combination of these constructions will give $c(8,0)$ which should also be possible by $p-q = 0 \pmod 8$. This requires an additional construction based on the existence of an eight dimensional, real, symmetric rep. of the algebra $\{\Gamma_A, \Gamma_B\} = 2\delta_{AB}$

Explicitly,

$$\Gamma_1 = \sigma_1 \times \sigma_3 \times \sigma_3 \times \sigma_3 \qquad \Gamma_5 = 1 \times \sigma_2 \times \sigma_1 \times \sigma_2$$

$$\Gamma_2 = 1 \times \sigma_1 \times \sigma_3 \times \sigma_3 \qquad \Gamma_6 = \sigma_2 \times \sigma_3 \times \sigma_1 \times \sigma_2$$

$$\Gamma_3 = 1 \times 1 \times \sigma_1 \times \sigma_3 \qquad \Gamma_7 = \sigma_2 \times \sigma_1 \times 1 \times \sigma_2$$

$$\Gamma_4 = 1 \times 1 \times 1 \times \sigma_1 \qquad \Gamma_8 = \sigma_2 \times \sigma_1 \times \sigma_2 \times \sigma_3$$

$$\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \Gamma_5 \Gamma_6 \Gamma_7 \Gamma_8 \equiv \Gamma_9 = -\sigma_3 \times \sigma_3 \times \sigma_3 \times \sigma_3$$

$$(\Gamma_9)^2 = +1 \qquad \{\Gamma_9, \Gamma_\lambda\} = 0$$

The construction $c(p,q) \to c(p+8,q)$ is:

$$P_i = P_i \times \Gamma_9$$

$$P_{p+1} = 1 \times \Gamma_1 \qquad Q_j = Q_j \times \Gamma_9$$

$$\vdots \qquad \vdots$$

$$P_{p+8} = 1 \times \Gamma_8$$

(The reason this doesn't work with $c(2,0)$ replacing $c(8,0)$ is that $(\Gamma_1\Gamma_2)^2 = (\sigma_1\sigma_3)^2 = (-i\sigma_2)^2 = -1$)

THE WEYL CONDITION

Independent of whether a real rep. exists, one can always impose the Weyl condition on the spinors

$$\left(\frac{1+\Gamma}{2}\right)\psi = 0 \quad \left(\text{or } \left(\frac{1-\Gamma}{2}\right)\psi = 0\right)$$

where $\Gamma = \Gamma_1 \cdots \Gamma_p \Gamma_{p+1} \cdots \Gamma_{p+q}\ (i)$

$$p+q = D = \text{even}$$

When necessary, the factor i is included so that $\Gamma^2 = +1$ (because $(P_\pm)^2 = P_\pm$ $P_+P_- = P_-P_+ = 0$ $P_\pm = \frac{1}{2}(1 \pm \Gamma)$). However, if the spinors are going to be <u>real</u> and a real rep. of the Clifford algebra exists, then the Weyl condition can only be imposed provided Γ can be defined without the i.

Now,

$$\Gamma^2 = (\Gamma_1 \cdots \Gamma_p)(\Gamma_{p+1} \cdots \Gamma_{p+q})(\Gamma_1 \cdots \Gamma_p)(\Gamma_{p+1} \cdots \Gamma_{p+q})$$

$$= (-1)^{pq}(\Gamma_1 \cdots \Gamma_p)(\Gamma_1 \cdots \Gamma_p)(\Gamma_{p+1} \cdots \Gamma_{p+q})(\Gamma_{p+1} \cdots \Gamma_{p+q})$$

$$= (-1)^{pq + \frac{1}{2}p(p-1) + \frac{1}{2}q(q-1)} (\Gamma_1 \cdots \Gamma_p)(\Gamma_p \cdots \Gamma_1) \cdot (\Gamma_{p+1} \cdots \Gamma_{p+q})(\Gamma_{p+q} \cdots \Gamma_{p+1})$$

$$= (-1)^{\frac{1}{2}(p^2+q^2+2pq-p-q)} (-1)^q$$

$$= (-1)^{\frac{1}{2}(p+q)(p+q-1) + q}$$

Therefore, both Weyl and Majorana conditions can be imposed provided,

$$(p+q)(p+q-1) + 2q = 0 \pmod 4$$

COUNTING SPINOR DEGREES OF FREEDOM

When constructing supersymmetric field theories we need to match bosonic with fermionic degrees of freedom. The number of states of a spinor satisfying the massless Dirac equation,

$$\Gamma \cdot \partial \psi = 0$$

with momentum $P_M = (\underbrace{P, P, 0, 0, \ldots}_{D = \text{even components}})$

$$\Gamma \cdot \partial \psi \rightarrow (\Gamma_0 + \Gamma_1)\psi = 0$$

is just the dimension of the null space of the matrix $\Gamma_0 + \Gamma_1$. Since this is independent of the choice of representation we can use a particular one:

$$D = 2 \quad : \quad \Gamma_0 = i\sigma_2 \qquad \Gamma_1 = \sigma_1$$

$$D = D' + 2 \quad : \quad \Gamma_0 = 1 \times \Gamma_0' \qquad \Gamma_i = 1 \times \Gamma_i'$$

$$\Gamma_{i+1} = \sigma_1 \times \Gamma$$

$$\Gamma_{i+2} = \sigma_3 \times \Gamma$$

$$\Gamma = \Gamma_0' \Gamma_1' \cdots \Gamma_{D'-1}' \quad (i)$$

(When necessary so $\Gamma^2 = +1$)

This construction generates a rep. for all even dimensions. Now,

$$\Gamma_0 + \Gamma_1 = 1 \times 1 \times \cdots \times \underbrace{(i\sigma_2 + \sigma_1)}_{\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}}$$

So the null space is half the dimensionality of the original space or $\frac{1}{2} 2^{D/2}$

Depending on whether or not the spinors are real or the Weyl condition is imposed, we now have the following number of states:

Complex	$2 \times \frac{1}{2} 2^{D/2}$
Complex + Weyl	$\frac{1}{2} 2^{D/2}$
Real	$\frac{1}{2} 2^{D/2}$
Real + Weyl	$\frac{1}{4} 2^{D/2}$ (for $D>2$)

Below are some cases of physical interest:

$D = p+1$	Real	Weyl	# states
2	yes	yes	1
4	yes	no	2
6	no	yes	4
8	no	yes	8
10	yes	yes	8
	$p = 1, 3 \pmod 8$	$(p+1)p = 2 \pmod 4$	

SUSY Yang-Mills in 10 dimensions.

Any compact Y-M. group G of generators X_i

$[X_i, X_j] = i f_{ijk} X_k$ $\quad$ f_{ijk} totally antisymm.

$$v_M = v_{Mi} X_i \qquad \lambda = \lambda_i X_i$$

$M = 0, 1, 2 \cdots 9$

$$v_{MN} = \partial_M v_N - \partial_N v_M + ig[v_M, v_N]$$

$$D_M \lambda = \partial_M \lambda + ig[v_M, \lambda]$$

$$I = \int d^{10}x \, \mathrm{Tr}\left(-\frac{1}{4} v_{MN} v^{MN} - \frac{i}{2} \bar{\lambda} \Gamma^M D_M \lambda\right)$$

$\bar{\lambda} = \lambda \Gamma^0$ $\quad$ λ Majorana <u>and</u> Weyl spinor

Susy Transf. law

$$\begin{cases} \delta v_M = i \bar{\alpha} \Gamma_M \lambda \\ \delta \lambda = -\frac{1}{2} v_{MN} \Gamma^M \Gamma^N \alpha \end{cases}$$

Note: The "little group" of the vector v_M is $SO(8)$ giving 8 states.

By the previous page, the Majorana-Weyl spinor λ also has 8 states.

Dimensional reduction. Introduce

$$\{\alpha^i, \alpha^j\} = \{\beta^i, \beta^j\} = -2\delta^{ij}$$

$$[\alpha^i, \beta^j] = 0 \qquad \alpha^1\alpha^2 = -\alpha^3 \quad (\alpha^i)^2 = -1$$

$$[\alpha^i, \alpha^j] = -2\varepsilon^{ijk}\alpha^k \qquad \beta^1\beta^2 = -\beta^3 \quad (\beta^i)^2 = -1$$

Algebra of $SU(2)\times SU(2)$

$$[\beta^i, \beta^j] = -2\varepsilon^{ijk}\alpha^k$$

Majorana representation $\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}$

$$\Gamma^m = \gamma^m \times \begin{pmatrix} 1_4 & 0 \\ 0 & -1_4 \end{pmatrix} \qquad m = 0,1,2,3$$

$$\Gamma^{3+j} = 1_4 \times \beta^3 \begin{pmatrix} 0 & \alpha^j \\ \alpha^j & 0 \end{pmatrix} \qquad j = 1,2,3$$

$$\Gamma^{6+i} = \gamma_5 \times \begin{pmatrix} \beta^i & 0 \\ 0 & \beta_i \end{pmatrix} \qquad i = 1,2,3$$

$\beta^1 = \beta_1$, $\beta^2 = \beta_2$, $\beta^3 = -\beta_3$ or $\beta^i = \eta_i\beta_i$ (no sum)

$$\Gamma = \Gamma_0\Gamma_1 \cdots \Gamma_9 = 1_4 \times \begin{pmatrix} 0 & -\beta_3 \\ \beta_3 & 0 \end{pmatrix}$$

$$\Gamma^2 = 1_4 \times \begin{pmatrix} 1_4 & 0 \\ 0 & 1_4 \end{pmatrix}$$

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A Weyl-Majorana spinor has the form

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \beta_3\psi_1 \\ \beta_3\psi_2 \\ \beta_3\psi_3 \\ \beta_3\psi_4 \end{pmatrix}$$

where $\psi_1, \psi_2, \psi_3, \psi_4$ are four ordinary 4-comps Majorana spinors.

$4\times 4 = 16$ independent components.

Vector $v_M = (v_m, A_i, B_i)$ $\quad i = 1,2,3$.

Assume all fields independent of the coordinates $x^4 \cdots x^9$. One finds in 4 dimensions.

$$\int d^4x \, \mathrm{Tr}\Big\{ -\frac{1}{4} v_{mn} v^{mn} - \frac{1}{2}(D_m A^i)^2 - \frac{1}{2}(D_m B^i)^2$$

$$- \frac{i}{2}\bar{\lambda}_k \gamma^m D_m \lambda_k + \frac{g}{2}\bar{\lambda}_k \left[\left(\alpha^j_{ke} A_j + \gamma_5 \beta^j_{ke} B_j\right), \lambda_e \right]$$

$$+ \frac{g^2}{4}\left([A_i, A_j]^2 + [B_i, B_j]^2 + 2[A_i, B_j]^2 \right) \Big\}$$

$i, j = 1,2,3 \qquad k, \ell = 1,2,3,4 \qquad m, n = 0,1,2,3$

Clearly the theory is invariant under $SO(6) \sim SU(4)$

Observe that

$$[\Gamma^{3+i}, \Gamma^{3+j}] = -1_4 \times \begin{pmatrix} [\alpha^i, \alpha^j] & 0 \\ 0 & [\alpha^i, \alpha^j] \end{pmatrix}$$

$$= 2\, 1_4 \times \varepsilon^{ijk} \begin{pmatrix} \alpha^k & 0 \\ 0 & \alpha^k \end{pmatrix}$$

$$[\Gamma^{6+i}, \Gamma^{6+j}] = 2\, 1_4 \times \eta_j \varepsilon^{ijk} \begin{pmatrix} \beta^k & 0 \\ 0 & \beta^k \end{pmatrix}$$

$$[\Gamma^{3+i}, \Gamma^{6+j}] = 2\gamma_5 \times \beta^3 \begin{pmatrix} 0 & \alpha^i \beta_j \\ \alpha^i \beta^j & 0 \end{pmatrix}$$

One can take them as generators of $SO(6)$ on spinors.
Finally the $SO(6)$ transformations are

$$\begin{cases} \delta v_m = 0 \\ \delta \lambda_k = -\frac{1}{4}\left[\varepsilon^{ijs}\beta^s \Lambda_{ij} + \varepsilon^{ijs}\alpha^s \Lambda'_{ij} + \gamma_5 \alpha^i \beta^j \Lambda''_{ij}\right]_{ke} \lambda_e \\ \delta A_i = \Lambda'_{ij} A_j - \Lambda''_{ij} B_j \\ \delta B_i = \Lambda_{ij} B_j + \Lambda''_{ji} A_j \end{cases}$$

$$\Lambda_{ij} = -\Lambda_{ji} \qquad \Lambda'_{ij} = -\Lambda'_{ji}$$

The subgroup which commutes with parity is $SO(4)$ $(\Lambda''_{ij} = 0)$
$\sim SU(2) \times SU(2)$.

$$\beta^1 = i\sigma_2 \times \sigma_1 \quad , \quad \beta^2 = 1 \times i\sigma_2 \quad , \quad \beta^3 = +i\sigma_2 \times \sigma_3$$

$$\alpha^1 = \sigma_1 \times i\sigma_2 \quad , \quad \alpha^2 = \sigma_3 \times -i\sigma_2 \quad , \quad \alpha^3 = i\sigma_2 \times 1$$

$$\beta^3\alpha^1 = \sigma_3 \times \sigma_1$$
$$\beta^3\alpha^2 = \sigma_1 \times \sigma_1$$
$$\beta^3\alpha^3 = -1 \times \sigma_3$$

$$\Gamma^m = \gamma^m \times 1 \times 1 \times \sigma_3$$

$$\Gamma^4 = 1_4 \times \sigma_3 \times \sigma_1 \times \sigma_1$$
$$\Gamma^5 = 1_4 \times \sigma_1 \times \sigma_1 \times \sigma_1$$
$$\Gamma^6 = 1_4 \times -1 \times \sigma_3 \times \sigma_1$$
$$\Gamma^7 = \gamma_5 \times i\sigma_2 \times \sigma_1 \times 1$$
$$\Gamma^8 = \gamma_5 \times 1 \times i\sigma_2 \times 1$$
$$\Gamma^9 = \gamma_5 \times i\sigma_2 \times \sigma_3 \times \sigma_3$$

$$\gamma^0 = \sigma_1 \times i\sigma_2 \quad , \quad \gamma^1 = \sigma_2 \times \sigma_2 \quad , \quad \gamma^2 = 1 \times \sigma_1 \quad , \quad \gamma^3 = 1 \times \sigma_3$$

$$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = -\sigma_3 \times i\sigma_2$$

$$\Gamma = \Gamma_0\Gamma_1 \cdots \Gamma_9 = -1 \times 1 \times \sigma_2 \times \sigma_3 \times \sigma_2$$

[This repr. is different from the one obtained by $(1,1) \to (1+8,1)$ which is $\Gamma^0 = i\sigma_2 \times (P_1 \cdots P_8)$ $\Gamma^1 = \sigma_1 \times (P_1 \cdots P_8)$, $\Gamma^{1+\ell} = 1 \times P_\ell$, where P_ℓ are real symmetric 8×8 matrices ($\ell = 1 \cdots 8$). If one chooses P_ℓ as in my notes, one finds $\Gamma = \Gamma_0\Gamma_1 \cdots \Gamma_9 = \sigma_3 \times \sigma_3 \times \sigma_3 \times \sigma_3 \times \sigma_3$ diagonal].