SUPERSYMMETRY

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SUPERSYMMETRY = FERMI-BOSE SYMMETRY

SUSY

A, B, F, G real fields; ψ Majorana spinor anticommuting

$$L_{kin} = -\frac{1}{2}(\partial_a A)^2 - \frac{1}{2}(\partial_a B)^2 - \frac{i}{2}\bar{\psi}\gamma^a\partial_a\psi + \frac{1}{2}F^2 + \frac{1}{2}G^2$$

$$\partial_a = \frac{\partial}{\partial x^a}$$

Eq.s of motion

$$\Box A = 0$$

$$\Box B = 0$$

$$\gamma^a\partial_a\psi = 0$$

$$F = 0$$

$$G = 0$$

"Scalar" = "chiral" supermultiplet

$$\delta A = i\bar{\alpha}\psi$$

$$\delta B = i\bar{\alpha}\gamma_5\psi$$

$$\delta\psi = \left[\gamma^a\partial_a(A + \gamma_5 B) + F + \gamma_5 G\right]\alpha$$

$$\delta F = i\bar{\alpha}\gamma^a\partial_a\psi$$

$$\delta G = i\bar{\alpha}\gamma_5\gamma^a\partial_a\psi$$

α infinitesimal constant anticommuting spinorial parameter.

$\gamma^a \partial_a = \gamma \cdot \partial$

Lagrangian changes by a derivative

$$\delta \mathcal{L}_{kin} = -\partial_a A\, \partial^a(\delta A) - \partial_a B\, \partial^a(\delta B) - \frac{i}{2}\bar{\psi}\gamma\cdot\partial(\delta\psi)$$
$$-\frac{i}{2}\delta\bar{\psi}\gamma\cdot\partial\psi + F\delta F + G\delta G$$

$$-i\bar{\psi}\gamma\cdot\partial(\delta\psi) = -i\bar{\psi}\Box(A+\gamma_5 B)\alpha - i\bar{\psi}\gamma\cdot\partial(F+\gamma_5 G)\alpha$$
$$= -i\bar{\alpha}\Box(A+\gamma_5 B)\psi + i\bar{\alpha}\gamma\cdot\partial(F-\gamma_5 G)\psi$$

etc.

$$\delta\mathcal{L}_{kin} = \frac{i}{2}\bar{\alpha}\gamma\cdot\partial\left[\{F-\gamma_5 G - \gamma\cdot\partial(A+\gamma_5 B)\}\psi\right]$$
$$= \frac{i}{2}\partial^a\left[\bar{\alpha}\gamma_a\{F-\gamma_5 G-\gamma\cdot\partial(A+\gamma_5 B)\}\psi\right]$$

Action is invariant (up to boundary terms)

Equations of motion are covariant

$A' = F$, $B' = G$, $\psi' = \gamma\cdot\partial\psi$, $F' = \Box A$, $G' = \Box B$

Check eg.

$$\delta A' = \delta F = i\bar{\alpha}\gamma\cdot\partial\psi = i\bar{\alpha}\psi'$$

$$\delta\psi' = \gamma\cdot\partial(\delta\psi) = \gamma\cdot\partial\left[\gamma\cdot\partial(A+\gamma_5 B) + F + \gamma_5 G\right]\alpha$$
$$= \left[\Box(A+\gamma_5 B) + \gamma\cdot\partial(F+\gamma_5 G)\right]\alpha$$
$$= \left[F' + \gamma_5 G' + \gamma\cdot\partial(A'+\gamma_5 B')\right]\alpha$$

$$\delta F' = \Box\delta A = \Box i\bar{\alpha}\psi = i\bar{\alpha}\Box\psi = i\bar{\alpha}\gamma\cdot\partial\gamma\cdot\partial\psi$$
$$= i\bar{\alpha}\gamma\cdot\partial\psi'$$

GAMMA MATRIX PROPERTIES

$\gamma_a \gamma_b + \gamma_b \gamma_a = 2 \eta_{ab}$ $\qquad \eta_{00} = -1, \ \eta_{11} = \eta_{22} = \eta_{33} = 1$

$a, b \cdots = 0, 1, 2, 3$ $\qquad \eta_{ab} = 0, \ a \neq b$

γ_a real 4×4 matrices (Majorana representation)

Sixteen basic matrices γ_A $(A = 1, 2, \cdots 16)$ are 1, $\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$, γ_a, $\gamma_5 \gamma_a$, $\gamma_a \gamma_b$ $(a < b)$

All real

$\gamma_5 \gamma_a = -\gamma_a \gamma_5 \qquad \gamma_5^2 = -1 \qquad \gamma_5^T = -\gamma_5$ $\qquad$ T means transposed

For any 4×4 matrix Γ define the <u>adjoint</u> $\tilde{}$

$\tilde{\Gamma} = -\gamma_0 \Gamma^T \gamma_0$

For any two hermitean (real) anticommuting spinors α_1, α_2

$\bar{\alpha}_1 \Gamma \alpha_2 = \bar{\alpha}_2 \tilde{\Gamma} \alpha_1 \qquad \bar{\alpha} = \alpha \gamma^0$

$\tilde{\gamma}_A = \gamma_A$ $\qquad$ for $1, \gamma_5, \gamma_5 \gamma_a$

$\tilde{\gamma}_A = -\gamma_A$ $\qquad$ for $\gamma_a, \gamma_a \gamma_b$ $(a < b)$

The sixteen matrices γ_A have squares $= \pm 1$

For any γ_A define γ^A so that $\gamma_A \gamma^A = 1$ (no sum over A)

$\text{trace } \gamma_A \gamma^B = 4 \delta_A^{\ B}$

Fierz rearrangement formula

α, β, ψ anticommuting Majorana spinors

$$\boxed{(\bar{\alpha}\psi)\beta = -\frac{1}{4}\sum_A (\bar{\alpha}\gamma_A\beta)\gamma^A\psi}$$

Proof:

Any 4×4 matrix Γ can be written as

$$\Gamma = \sum_A c_A \gamma^A \quad ;$$

to find the coefficients c_A use

$$\text{trace}\, \gamma_B \gamma^A = 4\delta_B{}^A$$

So

$$c_B = \frac{1}{4}\,\text{tr}\,\Gamma\gamma_B \qquad \text{tr} = \text{trace}$$

Now take the matrix

$$\Gamma_{ij} = \beta_i \bar{\alpha}_j$$

$$\text{tr}\,\Gamma\gamma_B = \sum_{ij} \beta_i \bar{\alpha}_j (\gamma_B)_{ji} = -\bar{\alpha}\gamma_B\beta$$

spinors anticommute

Therefore

$$\beta_i \bar{\alpha}_j = -\frac{1}{4}\sum_A (\bar{\alpha}\gamma_A\beta)(\gamma^A)_{ij}$$

Multiply by ψ_j and sum over j

$$\beta_i \bar{\alpha}\psi = -\frac{1}{4}\sum_A (\bar{\alpha}\gamma_A\beta)(\gamma^A\psi)_i$$

QED.

SUSY ALGEBRA

$$\delta_2\delta_1 A = i\bar\alpha_1\delta_2\psi = i\bar\alpha_1\left[\gamma\cdot\partial(A+\gamma_5 B)+F+\gamma_5 G\right]\alpha_2$$

$$\bar\alpha_1\alpha_2 = \bar\alpha_2\alpha_1 \qquad \bar\alpha_1\gamma_5\alpha_2 = \bar\alpha_2\gamma_5\alpha_1$$

$$\bar\alpha_1\gamma_5\gamma_a\alpha_2 = \bar\alpha_2\gamma_5\gamma_a\alpha_1 \qquad \bar\alpha_1\gamma_a\alpha_2 = -\bar\alpha_2\gamma_a\alpha_1$$

$$(\delta_2\delta_1-\delta_1\delta_2)A = 2i\bar\alpha_1\gamma^a\alpha_2\,\partial_a A = \xi^a\partial_a A$$

$\xi^a = 2i\bar\alpha_1\gamma^a\alpha_2$ Infinitesimal translation

Check on spinor

$$\delta_2\delta_1\psi = \left[\gamma\cdot\partial(\delta_2 A+\gamma_5\delta_2 B)+\delta_2 F+\gamma_5\delta_2 G\right]\alpha_1$$

$$=\left[i\gamma\cdot\partial(\bar\alpha_2\psi)+i\gamma\cdot\partial\gamma_5(\bar\alpha_2\gamma_5\psi)+i\bar\alpha_2\gamma\cdot\partial\psi+i\bar\alpha_2\gamma_5\gamma\cdot\partial\psi\,\gamma_5\right]\alpha_1$$

Fierz rearrangement

$$i\bar\alpha_2\partial_a\psi\,\gamma^a\alpha_1 = -\tfrac{i}{4}\bar\alpha_2\gamma_A\alpha_1\,\gamma^a\gamma^A\partial_a\psi$$

$$-i\bar\alpha_2\gamma_5\partial_a\psi\,\gamma_5\gamma^a\alpha_1 = \tfrac{i}{4}\bar\alpha_2\gamma_A\alpha_1\,\gamma_5\gamma^a\gamma^A\gamma_5\partial_a\psi$$

$$i\bar\alpha_2\gamma^a\partial_a\psi\,\alpha_1 = -\tfrac{i}{4}\bar\alpha_2\gamma_A\alpha_1\,\gamma^A\gamma^a\partial_a\psi$$

$$i\bar\alpha_2\gamma_5\gamma^a\partial_a\psi\,\gamma_5\alpha_1 = -\tfrac{i}{4}\bar\alpha_2\gamma_A\alpha_1\,\gamma_5\gamma^A\gamma_5\gamma^a\partial_a\psi$$

Exchang 1 and 2 an subtract. Most terms cancel
One finds commutator =

$$-i(\bar\alpha_2\gamma_b\alpha_1)(\gamma^a\gamma^b+\gamma^b\gamma^a)\partial_a\psi = 2i\bar\alpha_1\gamma^a\alpha_2\,\partial_a\psi$$

COMMUTATOR OF TWO SUSY TRANSFORMATIONS IS A TRANSLATION

Notice relevance of auxiliary fields for "off shell" closure of the algebra.

Combination of two chiral multiplets to a chiral multiplet.

$$A = A_1 A_2 - B_1 B_2$$

$$B = A_1 B_2 + B_1 A_2$$

$$\psi = (A_1 - \gamma_5 B_1)\psi_2 + (A_2 - \gamma_5 B_2)\psi_1$$

$$F = F_1 A_2 + F_2 A_1 + G_1 B_2 + G_2 B_1 - i\bar{\psi}_1 \psi_2$$

$$G = G_1 A_2 + G_2 A_1 - F_1 B_2 - F_2 B_1 + i\bar{\psi}_1 \gamma_5 \psi_2$$

F component changes by a divergence and can be taken as a term in the Lagrangian Taking twice the same multiplet one has the mass term

$$L_m = m\left(FA + GB - \tfrac{i}{2}\bar{\psi}\psi\right)$$

Combining once more with the same multiplet one has the trilinear interaction term

$$L_g = g\left[F(A^2 - B^2) + 2GAB - i\bar{\psi}(A - \gamma_5 B)\psi\right]$$

Total Lagr. of the W. Z. model

$$L = L_{kin} + L_m + L_g$$

A possible linear term λF in the Lagr. can be transformed away by redefining the field $A \to A + c$ with a suitable constant c.
Observe that $A+c$, B, ψ, F, G form a chiral multiplet, just as A, B, ψ, F, G.

$$L = L_{kin} + L_m + L_g$$

Eq.s of motion for the auxiliary fields are

$$F + mA + g(A^2 - B^2) = 0$$

$$G + mB + 2gAB = 0$$

Eliminating from here the aux. fields in the Lagr. one obtains

$$L_{kin} + L_m + L_g \longrightarrow -\frac{1}{2}(\partial_a A)^2 - \frac{1}{2}(\partial_a B)^2 - \frac{i}{2}\bar{\psi}\gamma\cdot\partial\psi$$

$$-\frac{1}{2}m^2A^2 - \frac{1}{2}m^2B^2 - \frac{im}{2}\bar{\psi}\psi$$

$$-gmA(A^2+B^2) - \frac{g^2}{2}(A^2+B^2)^2 - ig\bar{\psi}(A-\gamma_5 B)\psi$$

Exact susy requires all masses to be equal and couplings to satisfy special relations. Lagr. is renormalizable by power counting. It turns out that it is renormalizable keeping the susy connections among physical parameters.

Conserved vector-spinor current

$$J^a = \left[\gamma\cdot\partial(A-\gamma_5 B)\right]\gamma^a\psi + m\gamma^a(A-\gamma_5 B)\psi$$

$$+ g\gamma^a(A-\gamma_5 B)^2\psi$$

$$\partial_a J^a = 0$$

check from Eq.s of motion. $\psi(\bar{\psi}\psi) = \gamma_5\psi(\bar{\psi}\gamma_5\psi)$

Exercise in Fierz rearrangement

Let $A = (\bar{\psi}\psi)\psi$, $B = (\bar{\psi}\gamma_5\psi)\gamma_5\psi$

$C = (\bar{\psi}\gamma_5\gamma_a\psi)\,\gamma_5\gamma^a\psi$

Prove the identities

$$A = -\frac{1}{4}A + \frac{1}{4}B - \frac{1}{4}C \quad \text{or} \quad -5A + B - C = 0$$

$$B = \frac{1}{4}A - \frac{1}{4}B - \frac{1}{4}C \quad \text{or} \quad A - 5B - C = 0$$

$$C = -A - B + \frac{1}{2}C \quad \text{or} \quad 2A + 2B + C = 0$$

Three homogeneous eq.s in three unknowns, however the determinant is

$$\begin{vmatrix} -5 & 1 & -1 \\ 1 & -5 & -1 \\ 2 & 2 & 1 \end{vmatrix} = 0$$

So one finds only two relations

$$A = B = -\frac{1}{4}C$$

$$\boxed{(\bar{\psi}\psi)\psi = (\bar{\psi}\gamma_5\psi)\gamma_5\psi = -\frac{1}{4}(\bar{\psi}\gamma_5\gamma_a\psi)\gamma_5\gamma^a\psi}$$

ALGEBRA

Majorana

Let there be a spinorial operator Q_i such that, for any field $\phi = (A, B, \psi, F, G)$

$$\delta\phi = [\phi, \bar{\alpha} Q]$$

Example $\delta A = i\bar{\alpha}\psi = [A, \bar{\alpha} Q]$

Then $\delta_2 \delta_1 \phi = \left[[\phi, \bar{\alpha}_1 Q], \bar{\alpha}_2 Q\right]$

$$(\delta_2\delta_1 - \delta_1\delta_2)\phi = \left[\phi, [\bar{\alpha}_1 Q, \bar{\alpha}_2 Q]\right]$$

But we know that

$$(\delta_2\delta_1 - \delta_1\delta_2)\phi = 2i\,\bar{\alpha}_1 \gamma^a \alpha_2\, \partial_a \phi$$
$$= -2\,\bar{\alpha}_1 \gamma^a \alpha_2\, [\phi, P_a]$$

since $\partial_a \phi = i[\phi, P_a]$ $\qquad H = P^0$

Therefore

$$\boxed{[\bar{\alpha}_1 Q, \bar{\alpha}_2 Q] = -2\,\bar{\alpha}_1 \gamma^a \alpha_2\, P_a}$$

or

$$\boxed{\{Q_i, \bar{Q}_j\} = -2\,(\gamma^a)_{ij} P_a}$$

Multiply on the right with γ_0 and trace

$$H = \frac{1}{4}\sum_i (Q_i)^2 \geq 0$$

EMMY NOETHER'S THEOREM

$$L(\phi_i, \partial_a \phi_i)$$

$$\delta L = \frac{\partial L}{\partial \phi_i}\delta\phi_i + \frac{\partial L}{\partial(\partial_a\phi_i)}\partial_a(\delta\phi_i)$$

$$= \left[\frac{\partial L}{\partial \phi_i} - \partial_a \frac{\partial L}{\partial(\partial_a\phi_i)}\right]\delta\phi_i + \partial_a\left[\frac{\partial L}{\partial(\partial_a\phi_i)}\delta\phi_i\right]$$

Applying this to the action, for arbitrary variations $\delta\phi_i$, one finds the eq.s of motion

$$\frac{\partial L}{\partial \phi_i} - \partial_a \frac{\partial L}{\partial(\partial_a\phi_i)} = 0$$

For particular variations one may have

$$\delta L = \partial_a X^a$$

Then the equations of motion imply a conserved current

$$\partial_a\left[\frac{\partial L}{\partial(\partial_a\phi_i)}\delta\phi_i - X^a\right] = 0$$

In particular, this is true for $X^a = 0$, case of an invariant Lagrangian.

Let us apply this to susy. X^a is not zero (see p. 2) <u>and</u> careful with anticommuting fields!

Simple example;
Let δ represent an infinitesimal translation

$$\delta\phi_i = \xi^a \partial_a \phi_i \qquad \delta(\partial_b \phi_i) = \xi^a \partial_a (\partial_b \phi_i)$$

ξ^a constant vector

Then, if L does not depend explicitly on x,

$$\delta L = \xi^a \partial_a L$$

So

$$X^a = \xi^a L$$

Therefore the current

$$\frac{\partial L}{\partial(\partial_a \phi_i)} \xi^b \partial_b \phi_i - \xi^a L = -\xi^b T_b{}^a$$

is conserved (minus sign so that $T^{00} \geq 0$)

$$\partial_a T_b{}^a = 0$$

where

$$T_b{}^a = -\frac{\partial L}{\partial(\partial_a \phi_i)} \partial_b \phi_i + \delta_b{}^a L$$

This is the <u>canonical</u> energy momentum tensor
(not automatically symmetric)

$$\int T_b{}^a \, d\sigma_a = \int T_b{}^0 \, d_3x = P_b$$

P_b is the total four-momentum

What is X^a in the case of SUSY? We calculated on p2

$$\delta L_{kin} = \frac{i}{2}\partial_a\left[\bar\alpha\gamma^a\{F-\gamma_5 G-\gamma\cdot\partial(A+\gamma_5 B)\}\psi\right].$$

On the other hand

$$\delta L_m = im\,\partial_a\left[\bar\alpha\gamma^a(A-\gamma_5 B)\psi\right]$$

$$\delta L_g = ig\,\partial_a\left[\bar\alpha\gamma^a(A-\gamma_5 B)^2\psi\right].$$

These two formulas can be deduced immediately from the fact that L_m and L_g are F components of a supermultiplet and transform into the corresponding ψ components. Therefore

$$\delta L = \partial_a X^a$$

where

$$X^a = \frac{i}{2}\bar\alpha\gamma^a\{F-\gamma_5 G-\gamma\cdot\partial(A+\gamma_5 B)\}\psi$$
$$+im\,\bar\alpha\gamma^a(A-\gamma_5 B)\psi + ig\,\bar\alpha\gamma^a(A-\gamma_5 B)^2\psi.$$

On the other hand

$$\delta L = -\partial_a A\,\partial^a(\delta A) - \partial_a B\,\partial^a(\delta B) - \frac{i}{2}\bar\psi\gamma\cdot\partial(\delta\psi) - \frac{i}{2}\delta\bar\psi\gamma\cdot\partial\psi$$
$$+ \text{terms with no derivatives}$$
$$= \partial^a\left[-\partial_a A\,\delta A - \partial_a B\,\delta B - \frac{i}{2}\bar\psi\gamma_a\delta\psi\right]$$
$$+ \text{terms which vanish by the eq.s of motion.}$$

Therefore the combination

$$\partial_a A\,\delta A + \partial_a B\,\delta B + \frac{i}{2}\bar\psi\gamma_a\delta\psi + X_a$$

is conserved. Substituting from p. 1 and dropping the parameter $i\bar\alpha$ one finds the current given on p. 7

SUPERSPACE

Coordinates $\begin{cases} x^a & \text{commuting, vectorial} \\ \theta_i & \text{anticommuting, spinorial} \end{cases}$

Superfield is a function of x and θ

$$V(x^a, \theta_i).$$

Susy transformation

$$\delta V = \left[\alpha \frac{\partial}{\partial \theta} - i \bar{\alpha} \gamma^a \theta \, \partial_a \right] V$$

or equivalently

$$\delta \theta_i = \alpha_i \,, \quad \delta x^a = -i \bar{\alpha} \gamma^a \theta$$

Now

$$\delta_2 \delta_1 V = \left(\alpha_1 \frac{\partial}{\partial \theta} - i \bar{\alpha}_1 \gamma^a \theta \, \partial_a \right) \left(\alpha_2 \frac{\partial}{\partial \theta} - i \bar{\alpha}_2 \gamma^a \theta \, \partial_a \right) V$$

$$= i \alpha_1 \gamma^0 \gamma^a \alpha_2 \, \partial_a V + \cdots$$

The ... cancel in the commutator

$$(\delta_2 \delta_1 - \delta_1 \delta_2) V = 2 i \bar{\alpha}_1 \gamma^a \alpha_2 \, \partial_a V$$

Agrees with SUSY algebra.

$$V(x, \theta) = C(x) + i \bar{\theta} \chi(x) + \cdots\cdots + \theta_1 \theta_2 \theta_3 \theta_4 \, D(x)$$

$$(\theta_1)^2 = (\theta_2)^2 = (\theta_3)^4 = (\theta_4)^2$$

A superfield is a collection of ordinary fields which form a supermultiplet.

14 WEYL REPRESENTATION OF GAMMA MATRICES

TWO COMPONENT SPINORS

VAN DER WAERDEN NOTATION

$$\gamma_a = \begin{pmatrix} 0 & i\sigma_a \\ i\bar{\sigma}_a & 0 \end{pmatrix} \qquad \gamma_5 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$(\sigma_a)_{\alpha\dot{\beta}} = (1, \vec{\sigma}) \qquad (\bar{\sigma}_a)^{\dot{\alpha}\beta} = (1, -\vec{\sigma})$$

$$(\psi_\alpha)^* = \bar{\psi}_{\dot{\alpha}} \qquad (\bar{\psi}_{\dot{\alpha}})^* = \psi_\alpha$$

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta \qquad \bar{\psi}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}}$$

$$\varepsilon^{12} = -\varepsilon^{21} = \varepsilon^{\dot{1}\dot{2}} = -\varepsilon^{\dot{2}\dot{1}} = 1$$

$$\varepsilon^{\alpha\beta} = -\varepsilon_{\alpha\beta} \qquad \varepsilon^{\dot{\alpha}\dot{\beta}} = -\varepsilon_{\dot{\alpha}\dot{\beta}}$$

e.g. $\theta^\alpha\psi_\alpha = \varepsilon^{\alpha\beta}\theta_\beta\psi_\alpha = -\theta_\beta\varepsilon^{\beta\alpha}\psi_\alpha = -\theta_\beta\psi^\beta = \psi^\beta\theta_\beta$

$$\left(\frac{\partial}{\partial\theta^\alpha}\right)^* = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}$$

In the Weyl representation a Majorana spinor has the form

$$\psi = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}$$

$$\left[(\sigma_a)_{\alpha\dot{\beta}}\right]^* = (\bar{\sigma}_a)_{\dot{\alpha}\beta}$$

Useful identities

$$\sigma_a \bar{\sigma}^a = \bar{\sigma}_a \sigma^a = 4$$

$$\bar{\sigma}_a \sigma_b + \bar{\sigma}_b \sigma_a = \sigma_a \bar{\sigma}_b + \sigma_b \bar{\sigma}_a = -2\eta_{ab}$$

$$\sigma_a \bar{\sigma}_b \sigma^a = 2\sigma_b \qquad \bar{\sigma}_a \sigma_{bc} \bar{\sigma}^a = 0$$

where $$(\sigma_{ab})_\alpha{}^\beta = -\frac{1}{2}(\sigma_a \bar{\sigma}_b - \sigma_b \bar{\sigma}_a)_\alpha{}^\beta$$

$$(\bar{\psi}\chi)_4 = -i(\psi^\alpha \chi_\alpha + \bar{\chi}_{\dot\alpha} \bar{\psi}^{\dot\alpha})$$ Majorana spinors

$$(\bar{\psi}\gamma_5\chi)_4 = \psi^\alpha \chi_\alpha - \bar{\chi}_{\dot\alpha}\bar{\psi}^{\dot\alpha}$$

$$(\bar{\psi}\gamma_a\chi)_4 = \psi\sigma_a\bar{\chi} - \chi\sigma_a\bar{\psi}$$

$$(\bar{\psi}\gamma_5\gamma_a\chi)_4 = i(\psi\sigma_a\bar{\chi} + \chi\sigma_a\bar{\psi})$$

Notation $\psi\chi \equiv \psi^\alpha\chi_\alpha$, $\bar{\chi}\bar{\psi} \equiv \bar{\chi}_{\dot\alpha}\bar{\psi}^{\dot\alpha}$

SUPERSYMMETRY ALGEBRA

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2(\sigma^a)_{\alpha\dot\beta} P_a$$

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0$$

$$[Q_\alpha, P_a] = [\bar{Q}_{\dot\alpha}, P_a] = [P_a, P_b] = 0$$

INFINITESIMAL SUPERFIELD TRANSFORMATION OF PARAMETER ζ_α $\bar{\zeta}^{\dot\alpha}$

$$\delta V = \left[\zeta \frac{\partial}{\partial \theta} + \bar{\zeta} \frac{\partial}{\partial \bar{\theta}} + i (\theta \sigma^a \bar{\zeta} - \zeta \sigma^a \bar{\theta}) \partial_a \right] V$$

This is the same as on page 13, only written in two-comp. notation. Observe that

$$\left[\zeta \frac{\partial}{\partial \theta} + \bar{\zeta} \frac{\partial}{\partial \bar{\theta}} + i (\theta \sigma^a \bar{\zeta} - \zeta \sigma^a \bar{\theta}) \partial_a \,,\; \frac{\partial}{\partial \theta} + i \sigma^a \bar{\theta} \partial_a \right] = 0$$

Therefore the "supersymmetric derivative"

$$\mathcal{D}_\alpha = \frac{\partial}{\partial \theta^\alpha} + i (\sigma^a \bar{\theta})_\alpha \partial_a$$

satisfies

$$\delta \mathcal{D}_\alpha V = \mathcal{D}_\alpha \delta V.$$

Same is true for the conjugate

$$\bar{\mathcal{D}}_{\dot\alpha} = - \frac{\partial}{\partial \bar{\theta}^{\dot\alpha}} - i (\theta \sigma^a)_{\dot\alpha} \partial_a$$

Observe that

$$\{ \mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha} \} = - 2i (\sigma^a)_{\alpha\dot\alpha} \partial_a$$

$$\{ \mathcal{D}_\alpha, \mathcal{D}_\beta \} = \{ \bar{\mathcal{D}}_{\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta} \} = 0$$

$$[\mathcal{D}_\alpha, \partial_a] = [\bar{\mathcal{D}}_{\dot\alpha}, \partial_a] = [\partial_a, \partial_b] = 0$$

CHIRAL SUPERFIELD $\phi(x,\theta,\bar\theta)$

Satisfies the invariant constraint

$$\bar D_{\dot\alpha}\phi = 0$$

which reduces the number of independent components. ϕ must be complex.

Component fields can be defined as the values for $\theta=\bar\theta=0$ of

$$A=\sqrt{2}\,\phi \qquad \chi_\alpha = D_\alpha\phi \qquad F=-\frac{1}{2\sqrt{2}}D^\alpha D_\alpha\phi$$

(A and F are complex and correspond respectively to $\frac{1}{\sqrt{2}}(A-iB)$ and $\frac{1}{\sqrt{2}}(F+iG)$ of p. 1). Other derivatives are not independent. For instance

$$D_\beta D^\alpha D_\alpha\phi = 0$$

because

$$D_\alpha D_\beta D_\gamma = 0$$

(two indices must be equal);

$$\bar D_{\dot\beta}\, D^\alpha D_\alpha\phi = [\bar D_{\dot\beta}, D^\alpha D_\alpha]\phi = 4i\, D^\alpha(\sigma^a)_{\alpha\dot\beta}\,\partial_a\phi$$

$$= 4i(\sigma^a)_{\alpha\dot\beta}\,\partial_a\chi^\alpha$$

since

$$[\bar D_{\dot\beta}, D^\alpha D_\alpha] = 4i\, D^\alpha(\sigma^a)_{\alpha\dot\beta}\,\partial_a$$

If ϕ is chiral

$$\bar{D}_{\dot\alpha}\phi = 0 ,$$

then

$$\phi' = -\frac{1}{2\sqrt{2}} D^\alpha D_\alpha \phi$$

is antichiral

$$D_\alpha \phi' = 0 .$$

(ϕ' corresponds to the supermultiplet A', B' etc of p.2)

The eq.s of motion of the W.Z. model of p. 6 can be written compactly as

$$-\frac{1}{4}\bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha}\bar{\phi} + m\phi + 2g\phi^2 = 0 .$$

Observe that, for $\theta = \bar\theta = 0$, this gives

$$\bar{F} + mA + \sqrt{2}gA^2 = 0$$

which agrees with the eq.s for the auxiliary fields F and G on p. 7. The other equations of motion must come out correctly, since the superfield equation is supersymmetric. For instance apply D_α to the above eq. and use $D_\alpha\bar\phi = 0$ and

$$\left[D_\alpha, \bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha}\right] = -4i(\sigma^a)_{\alpha\dot\alpha}\bar{D}^{\dot\alpha}\partial_a .$$

Setting $\theta = \bar\theta = 0$, one finds

$$i(\sigma^a)_{\alpha\dot\alpha}\partial_a\bar\psi^{\dot\alpha} + m\psi_\alpha + 4gA\psi_\alpha = 0 .$$

Similarly, applying $D^\alpha D_\alpha$ to the superfield equation and using

$$\left[D^\alpha D_\alpha, \bar{D}_{\dot\alpha} \bar{D}^{\dot\alpha} \right] = 16 \Box + 8i (\sigma^a)_{\alpha\dot\alpha} \bar{D}^{\dot\alpha} D^\alpha \partial_a$$

$$= -16 \Box - 8i (\sigma^a)_{\alpha\dot\alpha} D^\alpha \bar{D}^{\dot\alpha} \partial_a$$

one finds, for $\theta = \bar\theta = 0$,

$$\Box \bar{A} + 8\sqrt{2}\, m F - 16 g \left(\psi^\alpha \psi_\alpha - 2\sqrt{2}\, A F \right) = 0 .$$

THE EQUATIONS OF MOTION FOR THE COMPONENT FIELDS ARE CONTAINED IN THE SUPERFIELD EQUATION OF MOTION.

THE TRANSFORMATION LAWS FOR THE COMPONENT FIELDS ARE CONTAINED IN THAT FOR THE SUPERFIELD. Observe that the transf. law for the superfield (top of p 16) can also be written

$$\delta V = \left[\xi^\alpha D_\alpha + \bar\xi_{\dot\alpha} \bar{D}^{\dot\alpha} + 2i \left(\theta \sigma^a \bar\xi - \xi \sigma^a \bar\theta \right) \partial_a \right] V .$$

Setting $\theta = \bar\theta = 0$, this gives, for the case of a chiral superfield ($V \to \phi$),

$$\delta A = \sqrt{2}\ \xi^\alpha \psi_\alpha . \qquad (\bar{D}\phi = 0)$$

On the other hand, apply D_β. We know that it commutes with the differential operator, so it can be applied directly to ϕ

$$\delta (D_\beta \phi) = \left[\xi^\alpha D_\alpha + \bar\xi_{\dot\alpha} \bar{D}^{\dot\alpha} \right] D_\beta \phi + \text{terms which vanish with } \theta, \bar\theta$$

$$= -\tfrac{1}{2} \xi_\beta D^\alpha D_\alpha \phi + 2i (\sigma^a)_{\beta\dot\alpha} \bar\xi^{\dot\alpha} \partial_a \phi + \text{"} ,$$

so,

$$\delta \psi_\beta = \sqrt{2} \left(\xi_\beta F + i (\sigma^a)_{\beta\dot\alpha} \bar\xi^{\dot\alpha} \partial_a A \right) .$$

Similarly for the auxiliary field F.

INTEGRATION OVER ANTI COMMUTING VARIABLES BEREZIN

Analogue of definite integral for ordinary functions (from $-\infty$ to $+\infty$) not the analogue of the primitive of a function. Consider, as an example functions of only one anticommuting variable θ

$$\theta^2 = 0$$

Such a function is always at most linear in θ

$$f(\theta) = a + \theta\beta = a - \beta\theta$$

In general there is no primitive $\varphi(\theta)$ such that

$$\frac{\partial}{\partial\theta}\varphi(\theta) = f(\theta)$$

simply because $\left(\frac{\partial}{\partial\theta}\right)^2 = 0$ implies that it should be

$$\frac{\partial f}{\partial\theta} = 0 \qquad \text{i.e. } \beta = 0 .$$

For functions $f(x)$ of a commuting variable x the definite integral is translationally invariant:

$$\int_{-\infty}^{\infty} f(x+a)\,dx = \int_{-\infty}^{\infty} f(x)\,dx .$$

We require the analogous property

$$\int_{\theta} f(\theta+\alpha) = \int_{\theta} f(\theta)$$

where α, like β and θ is fermionic (odd). So

$$\int_\theta (a + \theta\beta + \alpha\beta) = \int_\theta (a + \theta\beta) .$$

Assuming linearity

$$\int_\theta \alpha\beta = 0 .$$

This eq. is generalized by postulating that the integral of any constant vanishes. To obtain a non-trivial result, the integral of θ itself should not vanish, and is usually normalized to 1. Basic relations:

$$\int_\theta \text{constant} = 0 \quad , \quad \int_\theta \theta = 1 \quad .$$

Observe that then

$$\int_\theta f(\theta) = \frac{\partial}{\partial\theta} f(\theta) .$$

The delta function for anticommuting variables is

$$\delta(\theta - \theta') = \theta - \theta' . \qquad (\theta' \text{ is fermionic})$$

Indeed

$$\int_\theta (\theta - \theta')(a + \theta\beta) = a + \theta'\beta$$

Observe that, if β is a fermionic constant

$$\int_\theta \beta\theta = -\beta \int_\theta \theta = -\beta$$

corresponding to

$$\frac{\partial}{\partial\theta}(\beta\theta) = -\beta \frac{\partial}{\partial\theta}\theta = -\beta .$$

The integral of a derivative vanishes

$$\int_\theta \frac{\partial}{\partial\theta} f(\theta) = 0 . \qquad \left(\left(\frac{\partial}{\partial\theta}\right)^2 = 0 \right)$$

This implies the formula for integration by parts

$$\int_\theta f(\theta) \frac{\partial g(\theta)}{\partial\theta} = \mp \int_\theta \frac{\partial f(\theta)}{\partial\theta} g(\theta)$$

where the plus sign has to be used if $f(\theta)$ is itself fermionic.

For a function of several fermionic variables one applies the basic relations of p 24 successively for each variable. Observe that

$$\int_{\theta_1}\int_{\theta_2} f = -\int_{\theta_2}\int_{\theta_1} f$$

consistently with

$$\frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2} = -\frac{\partial}{\partial\theta_2}\frac{\partial}{\partial\theta_1} .$$

Example: Fourier transforms for functions of one fermionic variable θ.

Let $f(\theta) = a + \theta\beta$. The Fourier transform is (ρ fermionic)

$$\varphi(\rho) = \int_\theta e^{\rho\theta} f(\theta) = \int_\theta (1+\rho\theta)(a+\theta\beta) = -\rho a + \beta .$$

Now $$\int_\rho e^{-\rho\theta}\varphi(\rho) = \int_\rho (1-\rho\theta)(-\rho a+\beta)$$

$$= -\theta\beta - a = -f(\theta) \qquad \text{(inverse Fourier transform)}$$

This corresponds to

$$\int_\rho e^{-\rho(\theta'-\theta)} = \theta - \theta' = \delta(\theta-\theta')$$

Example: Fourier transform of a Gaussian, two fermionic variables θ_1 and θ_2,

$$\int d\theta_2 \int d\theta_1\, e^{\rho_1\theta_1+\rho_2\theta_2}\, e^{\sqrt{2}\,\theta_1\theta_2} = \int d\theta_2 \int d\theta_1 (1+\rho_1\theta_1+\rho_2\theta_2+2\rho_1\theta_1\rho_2\theta_2)\times$$

$$\times(1+\sqrt{2}\,\theta_1\theta_2) = \int d\theta_2 \int d\theta_1 \,(1+\rho_1\theta_1+\rho_2\theta_2+2\rho_1\theta_1\rho_2\theta_2+\sqrt{2}\,\theta_1\theta_2)$$

$$= -2\rho_1\rho_2 + \sqrt{2} = \sqrt{2}\,(1-\sqrt{2}\,\rho_1\rho_2) = \sqrt{2}\; e^{-\sqrt{2}\,\rho_1\rho_2}$$

REAL SUPERFIELD $V(x,\theta,\bar\theta)$, VECTOR SUPERMULTIPLET.
Standard definition of the components: they are the values for $\theta=\bar\theta=0$ of the superfields

$$C = V\ ,\quad \chi_\alpha = -i D_\alpha V\ ,\quad M = \frac{i}{4} D^\alpha D_\alpha V$$

$$v_{\alpha\dot\alpha} = -\frac{1}{2}[D_\alpha, \bar D_{\dot\alpha}] V\ ,\quad \lambda_\alpha = -\frac{i}{4} W_\alpha$$

$$D = \frac{1}{8} D^\alpha W_\alpha = \frac{1}{8} \bar D_{\dot\alpha} \bar W^{\dot\alpha} \quad \text{(real)}$$

where

$$W_\alpha = \bar D_{\dot\alpha} \bar D^{\dot\alpha} D_\alpha V\ ,\quad \bar D_{\dot\alpha} W_\alpha = 0 \quad \text{(chiral)}$$

Observe that, under the "gauge transformation"

$$V \to V + i(\Lambda - \bar\Lambda)\ ,\quad \bar D_{\dot\alpha}\Lambda = 0\ ,$$

the components C, χ, M transform so that one can transform them to zero*, while

$$v_{\alpha\dot\alpha} \to -\frac{1}{2}\left[D_\alpha \bar D_{\dot\alpha}(-i\bar\Lambda) - \bar D_{\dot\alpha} D_\alpha (i\Lambda)\right]$$

$$= \partial_{\alpha\dot\alpha}(\Lambda + \bar\Lambda) \quad \text{(a usual gauge transf.)}$$

and λ_α, D are invariant.

Under supersymmetry, the D component transforms by a divergence (check!) and can therefore be taken as a supersymmetric Lagrangian.

---

* This corresponds to a choice of gauge, usually called the Wess–Zumino gauge (non supersymmetric)

SUPERSPACE LAGRANGIAN FOR THE W-Z MODEL

For the kinetic part one can take the action

$$\sim \int d^4x \int d^2\theta \int d^2\bar\theta \;\; \phi\bar\phi \quad , \qquad \bar D_{\dot\alpha}\phi = D_\alpha \bar\phi = 0$$

Observe that this is

$$\sim \int d^4x \frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2}\frac{\partial}{\partial\bar\theta_1}\frac{\partial}{\partial\bar\theta_2}(\phi\bar\phi) \sim \int d^4x\, D^\alpha D_\alpha \bar D_{\dot\alpha}\bar D^{\dot\alpha}(\phi\bar\phi)$$

$$\sim \int d^4x\; D^\alpha \bar D_{\dot\alpha}\bar D^{\dot\alpha} D_\alpha(\phi\bar\phi) \sim \int d^4x\; (\phi\bar\phi)_D$$

The $\sim$ means equal up to a multiplicative constant and up to terms with x derivatives. These, with suitable boundary conditions, do not contribute to the action. The total Lagr. of the W-Z model is exactly

$$L(x) = (\phi\bar\phi)_D + \mathrm{Re}\left(\sqrt{2}\, m\phi^2 + \frac{4\sqrt{2}}{3} g\phi^3\right)_F$$

The D component of a real superfield is automatically real, the F component of a chiral superfield is complex and one must take the real part.

The kinetic term can be written in a way more convenient to derive the equations of motion by use of the identity

$$D^\alpha \bar D_{\dot\alpha}\bar D^{\dot\alpha} D_\alpha = \bar D_{\dot\alpha} D^\alpha D_\alpha \bar D^{\dot\alpha} = \frac{1}{2}\left\{D^\alpha D_\alpha, \bar D_{\dot\alpha}\bar D^{\dot\alpha}\right\} - 8\Box$$

$$\Box = \partial_a\partial^a = -\frac{1}{2}\partial_{\alpha\dot\alpha}\partial^{\alpha\dot\alpha}$$

Then we see that, for any real superfield V, the D component can be rewritten as

$$V_D = \frac{1}{8} D^\alpha \bar{D}_{\dot\alpha} \bar{D}^{\dot\alpha} D_\alpha V = \frac{1}{16} \{ D^\alpha D_\alpha, \bar{D}_{\dot\alpha} \bar{D}^{\dot\alpha} \} V - \Box V$$

$$= -\frac{1}{2\sqrt{2}} \mathrm{Re} \left(\bar{D}_{\dot\alpha} \bar{D}^{\dot\alpha} V \right)_F - \Box V_C$$

Observe that $\bar{D}_{\dot\alpha} \bar{D}^{\dot\alpha} V$ is a chiral superfield. The Lagrangian of the previous page becomes

$$L(x) = -\frac{1}{2\sqrt{2}} \mathrm{Re} \left(\phi \, \bar{D}_{\dot\alpha} \bar{D}^{\dot\alpha} \bar{\phi} \right)_F - \Box \left(\phi \bar{\phi} \right)_C$$

$$+ \mathrm{Re} \left(\sqrt{2} m \phi^2 + \frac{4\sqrt{2}}{3} g \phi^3 \right)_F$$

The term with the $\Box$ integrates to zero. Varying the superfield ϕ by $\delta\phi$ (and keeping $\bar\phi$ fixed) one finds immediately the superfield eq. of motion of page 18. Indeed

$$\delta \int L \, d^4x = -\frac{1}{2\sqrt{2}} \int d^4x \left(\delta\phi \, \bar{D}_{\dot\alpha} \bar{D}^{\dot\alpha} \bar{\phi} \right)_F$$

$$+ \frac{1}{2} \int d^4x \left(\sqrt{2} m 2 \delta\phi \, \phi + 4\sqrt{2} g \, \delta\phi \, \phi^2 \right)_F$$

The factor $\frac{1}{2}$ in front of the second term comes from taking the real part. For the first term there is no such factor because the kinetic term in $L(x)$ above is already real, after integration over x. Of course, we have used the fact that, if ϕ, X are chiral superfields and $\delta\phi$ is arbitrary

$$\int d^4x \left(\delta\phi \, X \right)_F = 0 \quad \text{implies} \quad X = 0 .$$

SUPERFIELD LAGRANGIAN FOR A VECTOR SUPERMULTIPLET.

In terms of the superfield $W_\alpha = \bar{D}_{\dot\alpha} \bar{D}^{\dot\alpha} D_\alpha V$ (see p. 24), which is chiral, one can write the kinetic Lagrangian

$$L(x) = \frac{1}{32\sqrt{2}} \left(W^\alpha W_\alpha \right)_F$$

This can be worked out, if one remembers the definition of the F components of a chiral superfield (p. 17).

$$L = -\frac{1}{128} D^\beta D_\beta (W^\alpha W_\alpha)$$

$$= +\frac{1}{128} (D^\alpha W_\alpha)^2 + \cdots \qquad = \frac{1}{2} D^2 + \cdots$$

where the field D is defined on p. 24. The dots denote the kinetic term for the spinor field λ and for the vector field of p. 24. The latter is expressed in terms of the field strength $F_{ab} = -F_{ba}$, which in spinor notation corresponds to a symmetric spinor (complex) $F_{\alpha\beta} = F_{\beta\alpha}$ $\propto D_\alpha W_\beta + D_\beta W_\alpha$. The result, in four-dimensional notation is (check!)

$$L = \frac{1}{2} D^2 - \frac{i}{2} \bar\lambda \gamma^\mu \partial \lambda - \frac{1}{4} F_{ab}^2$$

$F_{ab} = \partial_a v_b - \partial_b v_a$, λ a real four-component spinor.

SUSY QED

The electron is described by a Dirac spinor. One needs two chiral superfields Φ_+ and Φ_-

$$\bar{D}_{\dot\alpha}\Phi_+ = \bar{D}_{\dot\alpha}\Phi_- = 0$$

The Lagr. of supersymmetric QED is (for zero mass)

$$L(x) = \frac{1}{32\sqrt{2}}(W^\alpha W_\alpha)_F + (\bar\Phi_+ e^V \Phi_+ + \bar\Phi_- e^{-V}\Phi_-)_D$$

It is invariant under the gauge transformation

$$V \to V + i(\Lambda - \bar\Lambda) \qquad \bar{D}_{\dot\alpha}\Lambda = 0 \qquad \text{(see p. 24)}$$

$$\Phi_+ \to e^{-i\Lambda}\Phi_+ \qquad \Phi_- \to e^{i\Lambda}\Phi_-$$

$$\bar\Phi_+ \to e^{i\bar\Lambda}\bar\Phi_+ \qquad \bar\Phi_- \to e^{-i\bar\Lambda}\bar\Phi_-$$

One may add a mass term (gauge invariant)

$$L_m(x) \propto \mathrm{Re}(m\Phi_+\Phi_-)_F$$

Supersymmetric quantum electrodynamics is the simplest example of a supersymmetric gauge theory (abelian). It is physically unacceptable: the anomalous magnetic moment of the electron (or muon) cancels to zero to all orders (Ferrara and Remiddi). If susy is broken, this is no longer true, even for spontaneous breaking. A susy breaking ~ 100 GeV is consistent with $g-2$. More precise limits on susy breaking could be calculated[28] (a good thesis problem ?)

RECAP. OF YANG MILLS THEORY

Vector potential v_a in the adjoint representation of a compact Lie group K. Can be described by a matrix (Hermitean).

Infinitesimal gauge transformation

$$\delta v_a = i[K, v_a] - \partial_a K$$

K is the infinitesimal matrix in the adjoint repres.

Now

$$\delta(\partial_b v_a - \partial_a v_b) = i[K, \partial_b v_a - \partial_a v_b] + i[\partial_b K, v_a] - i[\partial_a K, v_b]$$

$$\delta[v_b, v_a] = i[K, [v_b, v_a]] - [\partial_b K, v_a] - [v_b, \partial_a K]$$

so, for the field strength

$$v_{ba} = \partial_b v_a - \partial_a v_b + i[v_b, v_a] \ ,$$

one finds

$$\delta v_{ba} = i[K, v_{ba}] .$$

K, v_a, v_{ba} are "Lie algebra valued".

Finite gauge transformation

$$v_a \to e^{iK} v_a e^{-iK} - \partial_a e^{iK} \ e^{-iK}$$

$$v_{ab} \to e^{iK} v_{ab} e^{-iK}$$

Invariant Lagrangian, up to a normalization constant,

$$L \propto \frac{1}{g^2} \operatorname{trace}(v_{ab}^2)$$

where g is the coupling constant.

INTERACTION WITH OTHER FIELDS

Let a field φ belong to a representation of the Yang Mills group K and its transformation law be

$$\delta\varphi = i\kappa\varphi$$

where κ denotes here the infinitesimal matrix in the appropriate representation. Then

$$\delta(\partial_a\varphi) = \partial_a(\delta\varphi) = i\kappa\,\partial_a\varphi + i\,\partial_a\kappa\,\varphi$$

$$\delta(v_a\varphi) = i[\kappa, v_a]\varphi - \partial_a\kappa\,\varphi + v_a\, i\kappa\varphi$$

so, for the covariant derivative

$$\mathcal{D}_a\varphi = \partial_a\varphi + i v_a\varphi$$

one finds

$$\delta(\mathcal{D}_a\varphi) = i\kappa\,\mathcal{D}_a\varphi$$

The covariant derivative transforms like φ. In the above, v_a is also taken as a matrix in the repres. appropriate to φ.

Finite gauge transf

$$\varphi \to e^{i\kappa}\varphi\ , \quad \mathcal{D}_a\varphi \to e^{i\kappa}\mathcal{D}_a\varphi\ .$$

The Lagr. with interaction is obtained from the Lagr. without interaction by replacing the ordinary x derivatives by the covariant derivatives.

For instance, for a scalar field φ in a complex representation one would take

$$L_\varphi = -\sum \overline{\mathcal{D}_a\varphi}\,\mathcal{D}_a\varphi \qquad \text{(sum over all comps)}$$

SUSY YANG MILLS

Let $V(x,\theta,\bar\theta)$ be a superfield valued in the Lie algebra of K. For instance, for $SU(n)$, one can take V to be an $n\times n$ Hermitean matrix (traceless). Let $\Lambda(x,\theta,\bar\theta)$ be a (traceless) complex $n\times n$ matrix which is a chiral superfield († means Hermitean conjug.)

$$\bar D_{\dot\alpha}\Lambda = 0 \qquad D_\alpha \Lambda^\dagger = 0$$

We take as gauge transformation (in analogy with p 28)

$$e^V \to e^{-i\Lambda^\dagger} e^V e^{i\Lambda}, \qquad e^{-V} \to e^{-i\Lambda} e^{-V} e^{i\Lambda^\dagger}$$

Expanding one finds

$$V \to V + i(\Lambda - \Lambda^\dagger) - \frac{1}{2}\left([\Lambda,\Lambda^\dagger] + i[\Lambda+\Lambda^\dagger, V]\right) + \dots$$

which exhibits the gauge transformation for the component fields. Under the gauge transf.

$$e^{-V} D_\alpha e^V \to e^{-i\Lambda}\left(e^{-V} D_\alpha e^V\right) e^{i\Lambda} + e^{-i\Lambda} D_\alpha e^{i\Lambda}$$

and

$$\bar D_{\dot\beta}\left(e^{-V} D_\alpha e^V\right) \to e^{-i\Lambda}\left[\bar D_{\dot\beta}\left(e^{-V} D_\alpha e^V\right)\right] e^{i\Lambda} - 2i e^{-i\Lambda}\partial_{\alpha\dot\beta} e^{i\Lambda}$$

(use the chirality of Λ). Applying another $\bar D$ to this, the last term is annihilated. Therefore the superfield

$$W_\alpha = \bar D_{\dot\beta}\bar D^{\dot\beta}\left(e^{-V} D_\alpha e^V\right) \text{ transforms as } W_\alpha \to e^{-i\Lambda} W_\alpha e^{i\Lambda}$$

Observe that it is chiral (and traceless)

$$\bar D_{\dot\beta} W_\alpha = 0$$

The trace$(W^\alpha W_\alpha)$ is gauge invariant and chiral.

Its F component can be taken as the Lagrangian for the supersymmetric Yang-Mills theory -

$$L \propto \mathrm{Re}\left(\mathrm{tr}\, W^\alpha W_\alpha\right)_F$$

In terms of components one finds

$$L \propto \mathrm{tr}\left(-\frac{1}{4} v_{ab}^2 - \frac{i}{2}\bar{\lambda}\gamma^a \mathcal{D}_a \lambda + \frac{1}{2} D^2\right)$$

$$v_{ab} = \partial_a v_b - \partial_b v_a + ig\,[v_a, v_b]$$

$$\mathcal{D}_a \lambda = \partial_a \lambda + ig\,[v_a, \lambda]\,.$$

Here we have used four-component Majorana spinor notation, all fields v_a, λ and D are in the adjoint repres. of the gauge group (for SU(N) they are described by $N \times N$ hermitean traceless matrices) and we have introduced the coupling constant g in its more familiar place by rescaling the fields. The vector-spinor current

$$J^c \propto \mathrm{tr}\left(-\frac{1}{4} v_{ab}\left[\gamma^a, \gamma^b\right]\gamma^c \lambda\right)$$

is conserved as a consequence of the eq.s of motion (check!). The eq. for the field D is simply $D = 0$.

THE YANG MILLS THEORY WITH ONE MAJORANA SPINOR IN THE ADJOINT REPRESENTATION IS SUPERSYMMETRIC, NO FURTHER FIELDS OR COUPLINGS ARE NEEDED.

(Does one have to go to the W-Z gauge to obtain the above component Lagrangian?)

Interaction with other fields can be introduced in analogy with the case of SUSY QED, p. 28
Let the chiral superfield ϕ_{+i} belong to a representation of the internal symmetry group, the index i specifying the various components. Let it transform under the internal gauge group as

$$\phi_+ \to e^{-i\Lambda}\phi_+ \qquad \text{(matrix multiplication)}.$$

Similarly

$$\bar{\phi}_- \to e^{i\Lambda^\dagger}\bar{\phi}_- \qquad (\bar{\phi}_- \text{ antichiral})$$

Then

$$\left(\bar{\phi}_+ e^{V}\phi_+ + \phi_- e^{-V}\bar{\phi}_-\right)_D$$

is gauge invariant and can be taken as the Lagrangian for the interaction, including the kinetic terms for ϕ_+ and ϕ_-. An invariant mass term would be proportional to

$$\mathrm{Re}\left\{m(\phi_-\phi_+)_F\right\}.$$

It is invariant because

$$\phi_- \to \phi_- e^{i\Lambda} \qquad (\text{just as } \bar{\phi}_+ \to \bar{\phi}_+ e^{i\Lambda^\dagger})$$

If ϕ_+ belongs to a real representation, then $\Lambda^T = -\Lambda$ and ϕ_- transforms like ϕ_+, so one can choose

$$\phi_- = \phi_+ = \phi$$

Then the kinetic Lagr. with gauge inv. coupling becomes simply $(\bar{\Phi} e^{V} \Phi)_D$ and the mass term $Re\{m(\Phi^2)_F\}$. In particular one can choose Φ to belong to the adjoint representation, which is real. In terms of component fields and in real (Majorana) notation, the Lagr. can be worked out to be $\propto$

$$tr\Big\{ -\frac{1}{2}(\mathcal{D}_a A)^2 - \frac{1}{2}(\mathcal{D}_a B)^2 - \frac{i}{2}\bar{\psi}\gamma\cdot\mathcal{D}\psi + \frac{1}{2}F^2 + \frac{1}{2}G^2$$

$$+ g\bar{\lambda}[A + \gamma_5 B, \psi] + igD[A,B] + m\left(FA + GB - \frac{i}{2}\bar{\psi}\psi\right)$$

$$+ \left(-\frac{1}{4}v_{ab}^2 - \frac{i}{2}\bar{\lambda}\gamma\cdot\mathcal{D}\lambda + \frac{1}{2}D^2\right)\Big\}$$

$$\mathcal{D}_a A = \partial_a A + ig[v_a, A] \text{ etc.}$$

Here we have added the kinetic Lagr. for the vector multiplet, from p. 32, and have used matrix notation for all fields, since they are all in the adjoint representation.

In spite of the presence of scalar fields and scalar self-couplings this theory is asymptotically free. In general, if the vector multiplet V interacts with n chiral multiplets like Φ, all belonging to the adjoint representation of $SU(N)$, the one-loop Callan-Symanzik function is given by

$$\beta = -\frac{g^3}{16\pi^2}(3 - n)N$$

and is negative for $n < 3$ (asymptotic freedom). No other couplings and Callan Symanzik functions come up, since there is 34 only one independent coupling constant g (by Susy). All couplings are related

Let us consider the special case $m=0$ and combine the two Majorana spinors λ and ψ to a Dirac spinor

$$\varphi = \frac{1}{\sqrt{2}}(\lambda + i\psi) \qquad \text{(and also } F \text{ and } G\text{)}$$

One can eliminate the auxiliary field D_λ through its equation of motion

$$D + ig[A,B] = 0 .$$

Then the Lagrangian of p 34 becomes

$$\mathrm{tr}\Big\{ -\frac{1}{4}v_{ab}^2 - \frac{1}{2}(D_a A)^2 - \frac{1}{2}(D_a B)^2 - \frac{i}{2}\bar{\varphi}\overleftrightarrow{D}_a\varphi$$

$$- ig\bar{\varphi}[A + \gamma_5 B, \varphi] + \frac{1}{2}g^2([A,B])^2 \Big\}$$

This theory is asymptotically free, since $n=1$. It has the invariance

$$\varphi \to e^{i\omega}\varphi \quad , \quad \varphi^* \to e^{-i\omega}\varphi^*$$

which rotates λ into ψ. This is a rigid $U(1)$ (ω is x-independent). (Actually Fayet has shown that this generalizes to a rigid $SU(2)$ invariance, which operates on the left and right-handed components). So there are two supersymmetries which can be rotated into each other. By this argument they are defined only "on shell" because there are no auxiliary fields: the susy algebra closes only by use of the equations of motion for the spinors. The conserved current is $J \propto$

$$\mathrm{tr}\Big(-\frac{1}{4}v_{ab}[\gamma^a,\gamma^b]\gamma^c\varphi + ig[A,B]^{35}\gamma_5\gamma^c\varphi - i\gamma\cdot D(A-\gamma_5 B)\gamma^c\varphi\Big)$$

For $n=3$, $\beta(g)$ vanishes (see p. 34) at one loop. In this theory there are four Majorana spinors $\lambda, \psi_1, \psi_2, \psi_3$ and six real scalars, $A_1 B_1, A_2 B_2, A_3 B_3$ in addition to the Yang Mills field v_a (after elimination of the auxiliary fields) This suggests a rigid $SO(4)$ invariance. Indeed there is such an invariance, provided one includes in the Lagrangian a term proportional to

$$\mathrm{Re}\left(g\, \varepsilon^{ijk}\, \mathrm{tr}\, \phi_i [\phi_j, \phi_k] \right)_F \qquad \varepsilon^{123} = 1 \text{ etc.}$$

which is permitted by gauge invariance and simple supersymmetry. $SO(4)$ invariance fixes the coefficient of this term. It then turns out that the Lags. has actually a larger $SU(4)$ invarian which operates on the left and right-handed components. The vector is an $SU(4)$ singlet, the spinors are in the 4 and the scalars are in the self dual $6 = \bar{6}$ (three complex components). A, B $SU(4)$ indices

$$S_{AB} = -S_{BA} \qquad S^*_{AB} = S^{AB} = \frac{1}{2}\varepsilon^{ABCD} S_{CD}$$

All this after elimination of all auxiliary fields, otherwise the $SO(4)$ and $SU(4)$ invariances are not manifest. For this theory $\beta(g)$ has been shown to vanish to two- and three-loops.

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The Lagrangian of the SU(4) susy theory is (in the Majorana form of Gliozzi, Scherk and Olive)

$$L = \mathrm{tr}\Big\{ -\frac{1}{4} v_{ab} v^{ab} - \frac{1}{2}(\mathcal{D}_a A_i)^2 - \frac{1}{2}(\mathcal{D}_a B_i)^2$$

$$+ \frac{g^2}{4}\left([A_i, A_j]^2 + [B_i, B_j]^2 + 2[A_i, B_j]^2 \right)$$

$$- \frac{i}{2}\bar{\lambda}_i \gamma\cdot\mathcal{D}\lambda_i + \frac{g}{2}\bar{\lambda}_k [(\alpha^i_{ke} A_j + \beta^j_{ke}\gamma^5 B_j), \lambda_e] \Big\}$$

Here the indices k, ℓ run from 1 to 4, the indices i, j from 1 to 3. The six matrices α^i and β^i satisfy

$$[\alpha^i, \beta^j] = 0 \qquad [\alpha^i, \alpha^j] = [\beta^i, \beta^j] = -2\delta^{ij}$$

and are real and antisymmetric.
Explicit representation:

$$\beta^1 = \begin{pmatrix} 0 & i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} \quad \beta^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \beta^3 = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}$$

$$\alpha^1 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} \quad \alpha^2 = \begin{pmatrix} 0 & -\sigma^3 \\ \sigma^3 & 0 \end{pmatrix} \quad \alpha^3 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}$$

$SO(6) \sim SU(4)$ invariance $\Lambda_{ij} = -\Lambda_{ji}$, $\Lambda'_{ij} = -\Lambda'_{ji}$ and $\tilde{\Lambda}_{ij}$ are 15 parameters.

$$\delta v_a = 0$$

$$\delta\lambda_k = \frac{1}{2}\left[\sigma^{ij}\Lambda_{ij} + \sigma'^{ij}\Lambda'_{ij} + \gamma_5 K^{ij}\tilde{\Lambda}_{ij}\right]_{ke}\lambda_e$$

$$\delta A_i = \Lambda'_{ij} A_j + \tilde{\Lambda}_{ij} B_j$$

$$\delta B_i = \Lambda_{ij} B_j + \tilde{\Lambda}_{ij} A_j$$

$$\sigma^{ij} = \frac{1}{4}[\beta^i, \beta^j] \qquad \sigma'^{ij} = \frac{1}{4}[\alpha^i, \alpha^j] \qquad K^{ij} = \frac{1}{4}\{\alpha^i, \beta^j\}$$

SPONTANEOUS BREAKING OF SUSY. The Fayet-Iliopoulos term (violates parity)

Take SUSY QED (p. 28) and add to the Lagrangian the term (ξ a constant)

$$\xi V_D = \xi D$$

This term is supersymmetric, since D changes by a derivative under susy. It is also gauge invariant, since D is gauge inv. (p. 24) (Note: it would not be gauge inv. for a susy Yang-Mills theory based on a semisimple gauge group. The Fayet-Iliopoulos "trick" works only for $U(1)$).

We are interested in the "potential" of the scalars V. We drop the kinetic terms and all terms involving spinors and the vector field. Then we obtain

$$-V = F_+\bar{F}_+ + F_-\bar{F}_- + m(F_+A_+ + \bar{F}_+\bar{A}_+ + F_-A_- + \bar{F}_-\bar{A}_-)$$
$$+ \tfrac{1}{2}D^2 + gD(\bar{A}_+A_+ - \bar{A}_-A_-) + \xi D$$

The eq.s of motion for the auxiliary fields are

$$\bar{F}_+ + mA_+ = 0 \qquad \bar{F}_- + mA_- = 0 \quad \text{and compl. conj.}$$
$$D + g(\bar{A}_+A_+ - \bar{A}_-A_-) + \xi = 0$$

Substituting one finds

$$V = F_+\bar{F}_+ + F_-\bar{F}_- + \tfrac{1}{2}D^2 \geq 0$$

In this expression the auxiliary fields are meant to be those functions of the physical fields obtained from their eq.s of motion, e.g.

$$F_+ = F_+(A) = -m\bar{A}_+ \quad , \quad D = D(A) = -g(\bar{A}_+A_+ - \bar{A}_-A_-) - \xi$$

etc. V is non negative, in agreement with p. 9.

in the tree approximation.

If V is positive at its minimum, Susy is broken. If $V=0$ at the minimum, Susy is exact; this can only happen if all aux. fields F_+, F_- and D vanish, but this is clearly impossible if m and ξ are both different from zero.

We vary independently $A_\pm$ and $\bar{A}_\pm$. Now

$$\frac{\partial V}{\partial A_+} = \bar{A}_+ (m^2 - gD) \qquad \frac{\partial V}{\partial A_-} = \bar{A}_- (m^2 + gD).$$

(plus complex conj.)

These both vanish in the following cases:

1) $A_+ = A_- = 0$. Then $D = -\xi$, $V = \frac{1}{2}\xi^2$.

2) $A_+ = 0$, $D = -\frac{m^2}{g}$, $\bar{A}_- A_- = -\frac{m^2}{g^2} - \frac{\xi}{g}$, $V = \frac{m^2}{g}\left(\xi - \frac{1}{2}\frac{m^2}{g}\right)$

3) $A_- = 0$, $D = \frac{m^2}{g}$, $\bar{A}_+ A_+ = -\frac{m^2}{g^2} + \frac{\xi}{g}$, $V = \frac{m^2}{g}\left(\xi - \frac{1}{2}\frac{m^2}{g}\right)$

2) can happen only for $-\xi g \geq m^2$, 3) only for $\xi g \geq m^2$, in which case 2) or respectively 3) gives the actual minimum. If $|\xi g| < m^2$, then the minimum is given by 1).

Let us consider in some detail case 1). All second derivatives of the potential vanish at the minimum, except for

$$\frac{\partial^2 V}{\partial A_+ \partial \bar{A}_+} = m^2 + g\xi \quad \text{and} \quad \frac{\partial^2 V}{\partial A_- \partial \bar{A}_-} = m^2 - g\xi$$

which give the square masses of the fields A_+ and A_-, since there is no mixing. It is easy to see

that the masses of the fields ψ_+ and ψ_- remain equal to m, while those of v_a and λ vanish. Susy is indeed broken, as confirmed by the mass splitting among A_+, ψ_+ and A_-, ψ_-. The masses satisfy the sum rule

$$\sum_{\text{Bosons}} m^2 = \sum_{\text{Fermions}} m^2$$

The spinor λ is massless and plays the role of "Goldstone fermion". Its Susy transformation law, which is of the form $\delta\lambda = \dots + D\gamma_5\alpha$ now acquires a constant term due to $\langle D \rangle = -\xi$ $\quad D = -\xi + D'$

$$\delta\lambda = \dots + D'\gamma_5\alpha - \xi\gamma_5\alpha$$

This is characteristic of a Goldstone particle (analogy with non-linear pion dynamics). Now consider case 2) (similarly for 3)). The field A_- acquires a non vanishing $\langle A_- \rangle \neq 0$. This means that in addition to Susy, gauge invariance is spontaneously broken. The Higgs effect occurs and the field v_a acquires a mass. There is mixing. A suitable combination of the spinors remains massless and becomes the Goldstone spinor. The masses are different but the above sum rule[40] is still satisfied. (Check!)

GENERAL THEORY OF n CHIRAL SUPERFIELDS

$\bar{D}_{\dot\alpha}\Phi_i = 0 \qquad i = 1, \dots n.$ Sum over repeated indices.

$$L = (\bar\Phi_i \Phi_i)_D + 2\,\mathrm{Re}\left[f(\Phi)\right]_F$$

↑ kinetic term ↑ mass terms and interactions

$f(\Phi) = a + b_i \Phi_i + \frac{1}{2} m_{ij}\Phi_i\Phi_j + \frac{1}{3!} g_{ijk}\Phi_i\Phi_j\Phi_k$

cubic polynomial (renormal. theory) indep. of $\bar\Phi_i$.

$$\left[f(\Phi)\right]_F \propto D^\alpha D_\alpha f(\Phi) = D^\alpha\left(\frac{\partial f}{\partial \Phi_i} D_\alpha \Phi_i\right)$$

$$= \frac{\partial^2 f}{\partial\Phi_i \partial\Phi_j} D^\alpha\Phi_j \cdot D_\alpha\Phi_i + \frac{\partial f}{\partial\Phi_i} D^\alpha D_\alpha \Phi_i \qquad \text{for } \theta=\bar\theta=0.$$

The scalar part of $\left[f(\Phi)\right]_F$ is $\frac{\partial f}{\partial A_i} F_i$.

The spinor mass matrix (A_i dependent, so it includes the Yukawa couplings) is $\propto \frac{\partial^2 f}{\partial A_i \partial A_j}$

For x independent fields the scalar part of the Lagrangian is

$$L_{\text{scalar}} = F_i \bar F_i + F_i \frac{\partial f}{\partial A_i} + \bar F_i \overline{\frac{\partial f}{\partial A_i}}$$

The auxiliary fields are given by their eq.s of motion

$$\bar F_i = -\frac{\partial f}{\partial A_i} \qquad F_i = -\overline{\frac{\partial f}{\partial A_i}}$$

Substituting and changing sign, one finds the

Tree approximation scalar potential

$$V = F_i \bar{F}_i = \frac{\partial f}{\partial A_i} \overline{\frac{\partial f}{\partial A_i}} \geq 0$$

If V vanishes for some A_i, which means that the eq.s

$$\frac{\partial f}{\partial A_i} = 0$$

have a solution, Susy is exact. If these equations have no solution and the minimum of V is reached for some A_i and is positive. Susy is spontaneously broken. Now

$$\frac{\partial V}{\partial A_j} = \frac{\partial^2 f}{\partial A_j \partial A_i} \overline{\frac{\partial f}{\partial A_i}} .$$

At the minimum this must vanish while, if Susy is broken, $\overline{\frac{\partial f}{\partial A_i}}$ does not. Therefore the mass matrix of the fermions $\left.\frac{\partial^2 f}{\partial A_j \partial A_i}\right|_{min.}$ must have a vanishing eigenvalue. The massless fermion is the Goldstone fermion (Goldstino). Actually, one can prove that, if

$$V \geq a > 0$$

then

$$\det \frac{\partial^2 f}{\partial A_i \partial A_j} \equiv 0$$

vanishes identically, not only at the minimum.

43 O'RAIFEARTAIGH - FAYET MODEL OF SPONTANEOUS SUSY BREAKING.

Three chiral superfields $\phi_0\ \phi_1\ \phi_2$

$$f = \lambda\phi_0 + m\phi_1\phi_2 + g\phi_1^2\phi_0 \qquad \lambda, m, g \neq 0 \text{ real}$$

$$-\bar{F}_0 = \frac{\partial f}{\partial A_0} = \lambda + gA_1^2$$

$$-\bar{F}_1 = \frac{\partial f}{\partial A_1} = mA_2 + 2gA_1A_0 \quad ; \quad \frac{\partial^2 f}{\partial A_i \partial A_j} = \begin{pmatrix} 0 & 2gA_1 & 0 \\ 2gA_1 & 2gA_0 & m \\ 0 & m & 0 \end{pmatrix}$$

$$-\bar{F}_2 = \frac{\partial f}{\partial A_2} = mA_1$$

$\det \frac{\partial^2 f}{\partial A_i \partial A_j} \equiv 0$

Clearly $\bar{F}_2$ and $\bar{F}_0$ cannot vanish simultaneously. Susy is spontaneously broken.

$$V = \underbrace{|\lambda + gA_1^2|^2 + m^2|A_1|^2}_{\tilde{V}} + |mA_2 + 2gA_1A_0|^2$$

Minimum for $mA_2 + 2gA_1A_0 = 0$ since this leaves A_1 free. Minimize the rest $\tilde{V}$.

$$\frac{\partial \tilde{V}}{\partial A_1} = 2g\lambda A_1 + 2g^2 \bar{A}_1 A_1 \bar{A}_1 + m^2 \bar{A}_1 = 0$$

$$\frac{\partial \tilde{V}}{\partial \bar{A}_1} = 2g\lambda \bar{A}_1 + 2g^2 A_1 A_1 \bar{A}_1 + m^2 A_1 = 0$$

Add $\quad (m^2 + 2g\lambda + 2g^2 A_1 \bar{A}_1)(A_1 + \bar{A}_1) = 0$

Subtract $\quad (-m^2 + 2g\lambda - 2g^2 A_1 \bar{A}_1)(A_1 - \bar{A}_1) = 0$

Three possible solutions:

1) $A_1 = \bar{A}_1 = 0 \qquad \tilde{V}_{1)} = V_{1)} = \lambda^2$

cases 2) and 3) on next page.

2) $A_1 = \bar{A}_1 \neq 0$ real ; $m^2 + 2g\lambda + 2g^2 A_1 \bar{A}_1 = 0$

$$A_1^2 = -\frac{m^2}{2g^2} - \frac{\lambda}{g} > 0 \quad ; \quad V_{2)} = -\frac{m^4}{4g^2} - \frac{\lambda m^2}{g}$$

This case is realized if $-\frac{\lambda}{g} > \frac{m^2}{2g^2}$, which requires λ and g to have opposite sign

3) $A_1 = -\bar{A}_1 \neq 0$ imaginary ; $-m^2 + 2g\lambda - 2g^2 A_1 \bar{A}_1 = 0$

$$A_1 \bar{A}_1 = -A_1^2 = -\frac{m^2}{2g^2} + \frac{\lambda}{g} > 0 \quad ; \quad V_{3)} = -\frac{m^4}{4g^2} + \frac{\lambda m^2}{g}$$

This case is realized if $\frac{\lambda}{g} > \frac{m^2}{2g^2}$

Observe that

$$V_{1)} - V_{2)} = \left(\lambda + \frac{m^2}{2g}\right)^2 \quad ; \quad V_{1)} - V_{3)} = \left(\lambda - \frac{m^2}{2g}\right)^2$$

Conclusion :

If $\frac{m^2}{2g^2} < -\frac{\lambda}{g}$, 2) is possible and gives the minimum ; 3) is not possible.

If $\frac{m^2}{2g^2} < \frac{\lambda}{g}$, 3) is possible and gives the minimum ; 2) is not possible

If $\left|\frac{\lambda}{g}\right| < \frac{m^2}{2g^2}$, neither 2) nor 3) are possible and 1) is the minimum.

In case 1) $A_1 = \bar{A}_1 = 0$ so $A_2 = 0$. A_0 is <u>arbitrary</u>

There is a valley along which V takes the same value, which is the minimum.

In cases 2) and 3) there is also a valley of minima given by the eq. $m A_2 + 2g A_1 A_0 = 0$ <u>with the appropriate value of A_1</u>.

Calculate masses and check the sum rule of p.40 for this model.

45 SCALAR POTENTIAL FOR A SUSY GAUGE THEORY

For a simple gauge group the tree approximation Lagr. at the bottom of p 41 must be complemented by

$$\frac{1}{2}(D^a)^2 + g\, D^a(\bar{A}\, T^a A) \qquad \text{(sum over } a\text{)}$$

The first term comes from the kinetic part of the Yang-Mills Lagr., the second from the interaction. D^a is in the adjoint repres. of the group, A in some repr. in which the matrices T^a represent the group generators. For a semi-simple gauge group one has a sum of terms like the above. If the gauge group contains $U(1)$ factors, each such factors contributes a term

$$\frac{1}{2} D^2 + g_1 D(\bar{A} Y A) + \xi D$$

where g_1 is the $U(1)$ coupling constant and Y the $U(1)$ charge of the scalars (diagonal matrix). The Fayet-Iliopoulos term ξD is allowed only for $U(1)$ factors. For simple factors it would break gauge inv. and spoil the renormalizability of the theory.
Eliminating the fields D^a and D through their eq.s of motion

$$D^a + g\, \bar{A} T^a A = 0 \quad , \quad D + g_1 \bar{A} Y A + \xi = 0$$

changing the sign (Lagr. $\to$ Hamiltonian) and adding to the potential of page 42 one finds

$$V = \frac{\partial f}{\partial A_i}\frac{\partial \bar{f}}{\partial A_i} + \sum \frac{1}{2} g^2 (\bar{A} T^a A)^2 + \sum \frac{1}{2}(g \bar{A} Y A + \xi)^2 \geq 0$$

the general form of the tree potential for a susy gauge theory.

Observe that, if the chiral part $\left|\frac{\partial f}{\partial A_i}\right|^2$ already breaks susy, which means that its minimum corresponds to a non-vanishing value of the potential, the additional terms due to the gauge interaction will not change this situation, since they are positive*. So, an O'Raifeartaigh type susy breaking will be effective in a gauge theory as well. On the other hand, suppose that the chiral part of the potential vanishes for some value of the A_i. Susy can still be broken by the gauge terms, in particular through the ξ's. However, in some cases things conspire to reestablish susy even for non vanishing ξ. This is well illustrated by a model invented by Julius Wess, which is a generalization of the Fayet-Il. model. In addition to the chiral fields ϕ_+ and ϕ_- of charges $+1$ and -1, Wess takes a neutral ϕ.

$$L = L_{kin}(\text{vector}) + \left(\bar{\phi}_+ e^V \phi_+ + \bar{\phi}_- e^{-V} \phi_- + \bar{\phi}\phi + \xi V\right)_D$$

$$+ 2\,\mathrm{Re}\left(m\phi_+\phi_- + f\phi_+\phi_-\phi + \lambda\phi\right)_F$$

This Lagr. gives a scalar potential

$$U = F\bar{F} + F_+\bar{F}_+ + F_-\bar{F}_- + \frac{1}{2}D^2$$

where $\quad -\bar{F} = \lambda + f A_+ A_- \qquad D = \bar{A}_+ A_+ - \bar{A}_- A_- + \xi$

$$-\bar{F}_+ = A_-(m + fA)$$

$$-\bar{F}_- = A_+(m + fA)$$

The chiral part of the potential is invariant under the $U(1)$ group $\quad A_\pm \to e^{\pm i\varphi} A_\pm \qquad A \to A$

---

* provided the minimum is still reached for a finite value of the fields

equations

On the other hand the $\bar{F}=0$ are also invariant under this transformation for complex φ, for instance for $\varphi = -i\alpha$ (α real)

$$A_\pm \to e^{\pm\alpha} A_\pm \qquad A \to A .$$

This is so because the $\bar{F}$ come from a superpotential which is actually invariant under the above transf. since it cannot contain the complex conjugate fields. One can solve the eq.s $\bar{F}=0$ (p. 46) by choosing $m + fA = 0$, $\lambda + f A_+ A_- = 0$. The above transform. changes a solution into another solution, a point where the chiral part of the potential vanishes into another zero. While the $F, \bar{F}$ remain zero D changes into

$$D(\alpha) = e^{\alpha} \bar{A}_+ A_+ - e^{-\alpha} \bar{A}_- A_- + \xi$$

Clearly one can choose α so that $D(\alpha) = 0$. So there is a solution where the total potential vanishes for any ξ. This kind of argument, which uses the invariance of the superpotential under the complex extension of the group can be applied to semisimple groups as well. How does the Fayet-Iliopoulos model escape this? For case 1) (p. 39) this is clear. The solution $A_+ = A_- = 0$ is a fixed point of the transf $A_\pm \to e^{\pm\alpha} A_\pm$ and no new solutions are generated, D is indep. of α. Exercise: what happens in cases 2) and 3) of p. 39?

Homework assigned till now

1. Complete the calculation at the end of p.2
   $\delta B' = \cdots$, $\delta G' = \cdots$.

2. Find a formula for the space components of P_a in terms of Q_i (see p. 9).

3. Construct the energy momentum tensor for the Lagrangian $L = -\frac{1}{2}(\partial_a A)^2 - \frac{1}{2} m^2 A^2 + f(A)$. (see p. 11)

4. Check δL_{kin} of p.2.

5. (optional) Check the conservation equation for the current at the end of p.7 using the eqs. of motion.

6. Show that, up to terms which integrate to zero, the proportionality holds

$$\{D^2, \bar{D}^2\}(\Phi \bar{\Phi}) \propto L_{kin}$$

where Φ is a chiral superfield and L_{kin} is given on p 1.

7. Show that, for a vector supermultiplet, the proportionality holds

$$\mathrm{Re}(W^\alpha W_\alpha)_F \propto L_{kin}$$

where W_α is defined on p. 27 and L_{kin} is given at the bottom of that page.

8. For the O'Raifeartaigh model (p. 43, 44) compute the masses and show that the satisfy the same sum rule given on p. 40 for the Fayet-Iliopoulos model.

FEYNMAN RULES BY FUNCTIONAL METHODS ; EX. SELF INTERACTING SCALAR FIELD.

$$L = -\frac{1}{2}(\partial_a A)^2 - \frac{1}{2}m^2A^2 + L_I(A) = L_0 + L_I$$

$$Z[J] = N\int \mathcal{D}A \; e^{i\int L\,dx + i\int JA\,dx}$$ Generating functional.

$$Z_0[J] = N\int \mathcal{D}A \; e^{i\int L_0\,dx + i\int JA\,dx}$$

$$Z[J] = e^{i\int L_I\left[\frac{1}{i}\frac{\delta}{\delta J}\right]dx} Z_0[J]$$

To evaluate Z_0 observe that

$$\int L_0 + \int JA = \frac{1}{2}\int A(\Box - m^2)A + \int JA = \frac{1}{2}\int B^2 - \frac{1}{2}\int J(\Box - m^2)^{-1}J$$

where $$B = (\Box - m^2)^{1/2}A + (\Box - m^2)^{-1/2}J$$

Changing the integration variable from A to B and choosing suitably the normalization factor N one has

$$Z_0[J] = e^{-\frac{i}{2}\int J(\Box - m^2)^{-1}J}$$

$(\Box - m^2)^{-1} = -(p^2 + m^2)^{-1}$ is the free propagator and should be chosen with the appropriate $i\varepsilon$.
An alternative derivation can be given by observing that Z satisfies the functional differential eq.

$$\left\{(\Box - m^2)\frac{1}{i}\frac{\delta}{\delta J} + L_I'\left(\frac{1}{i}\frac{\delta}{\delta J}\right)\right\} Z = -JZ\,; \quad L_I'(A) = \frac{\partial L_I}{\partial A}$$

and Z_0 the corresponding eq. with $L_I = 0$.
Euclidean $t = -i\tau$, must agree with sign of $i\varepsilon$.

CHIRAL SCALAR SUPERFIELD. FREE ACTION

$$\int d^4x \frac{d^4\theta}{4} \bar\phi\phi + \left[\int d^4x \frac{d^2\theta}{2}\left(\frac{m}{2}\phi^2 + J\phi\right) + c.c.\right]$$

$$= \int d^4x \frac{d^4\theta}{4}\left[\bar\phi\phi - \frac{m}{2}\phi\left(-\frac{D^2}{4\Box}\phi\right) - \frac{m}{2}\bar\phi\left(-\frac{\bar D^2}{4\Box}\bar\phi\right)\right.$$

$$\left. + J\left(-\frac{D^2}{4\Box}\phi\right) + \bar J\left(-\frac{\bar D^2}{4\Box}\bar\phi\right)\right]$$

$$= \int d^4x \frac{d^4\theta}{4}\left[\frac{1}{2}(\phi\ \bar\phi)\begin{pmatrix} \frac{mD^2}{4\Box} & 1 \\ 1 & \frac{m\bar D^2}{4\Box}\end{pmatrix}\begin{pmatrix}\phi\\ \bar\phi\end{pmatrix} + (\phi\ \bar\phi)\begin{pmatrix} -\frac{D^2}{4\Box}J \\ -\frac{\bar D^2}{4\Box}\bar J\end{pmatrix}\right]$$

$$\equiv \int d^4x \frac{d^4\theta}{4}\left(\frac{1}{2}\Psi^T M\Psi + \Psi^T K\right) \qquad \bar D_{\dot\alpha}\phi = \bar D_{\dot\alpha} J = 0$$

We have used $\int d^4\theta \equiv \int d^2\theta \int \overline{d^2\theta}$

$$\int d^4x \int d^2\theta = \int d^4x \int d\theta_1 d\theta_2 = \int d^4x\left(-\frac{1}{2}D^\alpha D_\alpha\right) = \int d^4x\left(-\frac{1}{2}D^2\right)$$

$$\int d^4x \int \overline{d^2\theta} = \int d^4x \int d\bar\theta_2 d\bar\theta_1 = \int d^4x\left(-\frac{1}{2}\bar D_{\dot\alpha}\bar D^{\dot\alpha}\right) = \int d^4x\left(-\frac{1}{2}\bar D^2\right)$$

The eq. at the top of p. 19, applied to a chiral superfield gives $\bar D^2 D^2\phi = 16\,\Box\phi$. Therefore, if χ is also chiral, $\bar D\chi = 0$,

$$-\int d^4x\, d^4\theta\, \chi\frac{D^2}{8\Box}\phi = -\int d^4x \frac{D^2\bar D^2}{4}\left(\chi\frac{D^2}{8\Box}\phi\right) = -\int d^4x \frac{D^2}{2}(\chi\phi)$$

$$= \int d^4x\, d^2\theta\, \chi\phi$$

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EUCLIDEAN x-SPACE GENERATING FUNCTIONAL.

$$Z_0(J) = e^{-\frac{1}{2}\int d^4x \frac{d^4\theta}{4} K^T M^{-1} K}$$

$$M^{-1} = \begin{pmatrix} -\frac{m}{4}\bar{D}^2/(\Box - m^2) & 1 + m^2\bar{D}^2 D^2/16\Box(\Box - m^2) \\ 1 + m^2 D^2\bar{D}^2/16\Box(\Box - m^2) & -\frac{m}{4}D^2/(\Box - m^2) \end{pmatrix}$$

Check, using the eq. on top of p. 19. Then

$$-\frac{1}{2}\int d^4x \frac{d^4\theta}{4} K^T M^{-1} K = \int d^4x \frac{d^4\theta}{4}\left[\bar{J}(-\Box + m^2)^{-1} J\right.$$

$$\left. + \frac{1}{2} J\left(\frac{\frac{m}{4}D^2}{-\Box(-\Box + m^2)}\right)J + \frac{1}{2}\bar{J}\left(\frac{\frac{m}{4}\bar{D}^2}{-\Box(-\Box + m^2)}\right)\bar{J}\right]$$

This gives the free propagators

$$\underbrace{\phi\,\bar{\phi}} = \frac{1}{p^2 + m^2}\delta^4(\theta_{12}) \qquad \theta_{12} = \theta_1 - \theta_2$$

$$\underbrace{\phi\,\phi} = \frac{m}{p^2(p^2 + m^2)}\frac{1}{4}D^2(p, \theta_1)\,\delta^4(\theta_{12})$$

$$\underbrace{\bar{\phi}\,\bar{\phi}} = \frac{m}{p^2(p^2 + m^2)}\frac{1}{4}\bar{D}^2(p, \theta_1)\,\delta^4(\theta_{12}) .$$

To find the vertices one uses the formula

$$Z[J] = e^{S_I\left(\frac{\delta}{\delta J}\right)} Z_0[J]$$

were $S_I(\phi)$ is the interaction part of the action.

Example. In the W-Z model

$$S_I(\phi) = \int d^4x \frac{d^2\theta}{2}\left(-\frac{1}{3!} g \phi^3\right) + c.c.$$

$$S_I\left(\frac{\delta}{\delta J}\right) = \int d^4x \frac{d^2\theta}{2}\left(-\frac{1}{3!} g \left(\frac{\delta}{\delta J}\right)^3\right) + c.c.$$

How do functional derivatives with respect to a chiral source operate? It must be ($z = x, \theta, \bar\theta$)

$$\frac{\delta}{\delta J(z')} \int d^4x\, d^2\theta\, J(z)\phi(z) = \int d^4x\, d^2\theta \frac{\delta J(z)}{\delta J(z')}\phi(z)$$

$$= \phi(z') = \int d^4x\, d^4\theta\, \delta^8(z - z')\phi(z)$$

where

$$\delta^8(z) \equiv \delta^4(x)\,\delta^4(\theta)$$

This implies

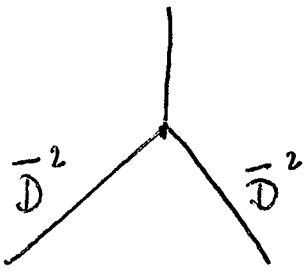
$$\frac{\delta J(z)}{\delta J(z')} = -\frac{1}{2}\bar D^2 \delta^8(z - z') .$$

Then, for instance

$$S_I\left(\frac{\delta}{\delta J}\right) J(z_1)J(z_2)J(z_3) = -\frac{g}{2\cdot 3!}\int d^4x_4 d^2\theta_4 \left(\frac{\delta}{\delta J(z_4)}\right)^3 J(z_1)J(z_2)J(z_3)$$

$$= -\frac{g}{2}\int d^4x_4 d^4\theta_4\, \delta^8(z_{14})\left[-\frac{1}{2}\bar D_2^2 \delta^8(z_{24})\right]\left[-\frac{1}{2}\bar D_3^2 \delta^8(z_{34})\right]$$

$\propto$



Two of the three lines carry a $\bar D^2$ for chiral fields D^2 for antichiral fields.

Each term in the effective action can be expressed as an integral over a single $d^4\theta$ (Grisaru, Roček and Siegel). The integrand is local in θ, although non-local in x (a function of many x's). This statement is true also with gauge fields. Consider a loop in a one-particle irreducible diagram:

$\delta^4(\theta_{12})\,\delta^4(\theta_{23})\cdots\delta^4(\theta_{n1})$ with $D, \bar{D}$ acting on the δ-functions. Integrate by parts to remove D_1 and D_2 from $\delta^4(\theta_{12})$ so they act on $\delta^4(\theta_{23})\cdots$ or on lines external to the loop, then integrate over θ_2 (or θ_1) removing $\delta^4(\theta_{12})$ and replacing θ_2 by θ_1. Continue until you have only one δ-function and the graph gives

$$\int d^4\theta_1\, d^4\hat{\theta}_i\; f(\theta_1, \hat{\theta}_i)\Big[\overbrace{D\cdots D}^{\text{some } D \text{ and some } \bar{D}}\delta^4(\theta_{n1})\Big]$$

$\hat{\theta}$ refer to θ of vertices not in the loop. $\theta_n = \theta_1$

Using the commut. relations one can reduce the product of D to 4 or fewer, but for fewer than 4 the expression vanishes for $\theta_n = \theta_1$. The only non vanishing term becomes $\left[D^2\bar{D}^2\delta^4(\theta_{n1})\right]_{\theta_n=\theta_1} = 16$. So the graph has the form

$$\int d^4\theta_1\, d^4\hat{\theta}_i\; \tilde{f}(\theta_1, \hat{\theta}_i)$$

and the loop has been contracted to a point in θ space. Repeat the procedure loop after loop. Final form

$$\int d^4x_1 \cdots d^4x_n\, d^4\theta\; F_1(x_1,\theta)\cdots F_n(x_n,\theta)\, G(x_1,\cdots x_n)$$

F's are products of superfields and their derivatives. G is translationally invariant.

The result at the bottom of page 53 has a number of interesting implications. All radiative corrections to the effective action are in the form of a bona fide $d^4\theta$ integral, without $\Box^{-1}$ or $(p^2)^{-1}$. Since mass terms and cubic interaction terms among scalars (cubic in superspace) are $d^2\theta$ integrals, this means that they are not modified by radiative corrections. Masses and scalar self couplings are not renormalized. The kinetic Lagrangian is given by a $d^4\theta$ integral (or D component, p. 25) and therefore is renormalized. So there is a wave function renormalization for each superfield. The wave function renormalization will affect masses and coupling constants, but only multiplicatively. If

$$\phi_{ren} = Z^{-1/2}\phi$$

one would have expected, from renorm. theory,

$$m_{ren} = Zm + \delta m, \qquad g_r = Z^{3/2}Z'g.$$

The above arguments show that

$$\delta m = 0 \qquad Z' = 1.$$

Susy by itself does not imply this result, and the deeper reason for it is not quite clear. The only satisfactory proofs are based on explicit considerations of perturbation theory as described above.

One can extract the effective potential from the effective action by assuming all fields (and therefore all superfields) to be x independent. This is equivalent to taking the momenta of all external legs of one-particle irreducible diagrams equal to zero. Since the effective action is the sum of the action plus a sum of terms like that at the bottom of p. 53, all these radiative corrections take the form

$$\int d^4\theta \; F_1(\theta) F_2(\theta) \cdots F_n(\theta) \int dx_1 \cdots dx_n \, G(x_1 \cdots x_n)$$

Therefore, the radiative corrections to the eff. pot. have the form

$$\frac{\bar\partial}{\partial\theta_2}\frac{\bar\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2}\frac{\partial}{\partial\theta_1} \mathcal{F}\left(\phi, \bar\phi, \frac{\partial}{\partial\theta^\alpha}\phi, \frac{\partial}{\partial\bar\theta_{\dot\alpha}}\bar\phi, \left(\frac{\partial}{\partial\theta}\right)^2\phi, \left(\frac{\bar\partial}{\partial\theta}\right)^2\bar\phi\right)$$

Dropping spinorial terms this becomes

$$L_{rad} = F_i \bar F_j \, h_{ij}(F, \bar F, A, \bar A)$$

to be added to the tree term, written as L_{scalar} on p. 41. All radiative corrections have a factor $F_i \bar F_j$, the function h_{ij} does not have a general form. One can try to minimize $L_{scalar} + L_{rad}$. Taking $\frac{\partial}{\partial \bar F_i}$ one finds

$$\bar F_i + \frac{\partial f}{\partial A_i} + \bar F_j h_{ij} + F_j \bar F_k \frac{\partial h_{jk}}{\partial \bar F_i} = 0$$

and taking $\frac{\partial}{\partial A_i}$

$$F_j \frac{\partial^2 f}{\partial A_i \partial A_j} + F_j \bar F_k \frac{\partial h_{jk}}{\partial A_i} = 0$$

If Susy is not broken at the tree level the eq.s

$$\frac{\partial f}{\partial A_i} = 0$$

have a solution $A_i = \mathring{A}_i$. Then the tree potential vanishes for $A_i = \mathring{A}_i$, because $F_i = 0$. One sees immediately that this solution is good in presence of rad. corr.s as well and that the full effective potential vanishes at that point. If Susy is not spontaneously broken at the tree level, it is not broken by radiative corrections. In terms of ordinary fields and Feynman diagrams the results of pp. 54–56 follow from cancellations of divergences between diagrams with loops of bosons and of fermions.

Observe that

$$\delta_{ij} + h_{ij}(0, 0, 0, 0)$$

is the matrix of wave-function renormalization constants (logarithmically divergent).

RECAP OF FADDE'EV POPOV GHOSTS.

The functional integral

$$Z = N \int e^{iS(v)} \mathcal{D}v$$

for a Yang-Mills action $S(v)$ is infinite because the action is constant in the directions of gauge transf.s. One wants to extract the infinite factor due to the integration over the gauge group. Let

$$v_a^g = g(v_a + g^{-1}\partial_a g)g^{-1}$$

be the gauge transformed of v_a by an element g of the gauge group, and let dg be the group-invariant measure

$$d(g_0 g) = dg .$$

Let

$$f(v) = 0$$

be a gauge fixing condition, for instance $f(v) = \partial_a v^a$. Define $\Delta_f(v)$ by

$$\Delta_f(v) \int \mathcal{D}g \; \delta(f(v^g)) = 1$$

where $\mathcal{D}g = \prod_x dg(x)$ and $\delta(\,)$ is the functional delta funct. One has

$$\Delta_f(v^g) = \Delta_f(v) .$$

Indeed

$$\frac{1}{\Delta_f(vg)} = \int \mathcal{D}g' \, \delta(f(v^{gg'})) = \int \mathcal{D}(gg') \, \delta(f(v^{gg'})) =$$

$$= \int \mathcal{D}g'' \, \delta(f(v^{g''})) = \frac{1}{\Delta_f(v)}$$

Insert $1 = \Delta_f \int \mathcal{D}g\, \delta(f(v^g))$ into the functional integral for Z

$$Z = N \int e^{iS(v)} \Delta_f(v)\, \delta(f(v^g))\, \mathcal{D}g\, \mathcal{D}v = \quad (v \to v^{g^{-1}})$$

$$= N \int e^{iS(v)} \Delta_f(v)\, \delta(f(v))\, \mathcal{D}g\, \mathcal{D}v$$

$$= N \int \mathcal{D}g \int e^{iS(v)} \Delta_f(v)\, \delta(f(v))\, \mathcal{D}v .$$

We have used the invariance of $S(v)$, $\Delta_f(v)$ and $\mathcal{D}v$ under gauge transf's on v. The group integration is isolated as a factor and can be chopped. We redefine

$$Z = N \int e^{iS(v)} \Delta_f(v)\, \delta(f(v))\, \mathcal{D}v .$$

Now evaluate $\Delta_f(v)$. Because of the δ-function it suffices to evaluate it for v satisfying $f(v) = 0$. So, take an infinitesimal transf

$$g = 1 + \Lambda \qquad dg = d\Lambda$$

$$f(v^g(x)) = f(v(x)) + \int dx'\, M_f(x, x')\, \Lambda(x')$$

$$M = \frac{\delta f(v)}{\delta \Lambda}$$

$$\Delta_f^{-1}(v) = \int \mathcal{D}\Lambda\, \delta(f(v^g)) = \int \mathcal{D}\Lambda\, \delta(M_f \Lambda) = \frac{1}{\det M_f}$$

$$\Delta_f(v) = \det M_f = \det \frac{\delta f(v)}{\delta \Lambda} \qquad \text{for } f(v) = 0 .$$

(Functional determinant!).

Conclusion:

$$Z = N \int e^{iS(v)} \det \frac{\delta f(v)}{\delta \Lambda}\, \delta(f(v))\, \mathcal{D}v .$$

This form of the functional is not convenient for perturbation theory. If $c(x)$, $c'(x)$ are anticommuting

$$\int e^{i\iint c' M c} \mathcal{D}c' \mathcal{D}c \propto \det M$$

Let us also consider gauge fixing conditions of the form $f(v) - f_0 = 0$ (this is just another function $f(v(x))$, namely $f(v(x)) - f_0(x)$). Then the redefined functional can be written as

$$Z_{f_0} = N \int e^{iS(v) + i\iint c' \frac{\delta f}{\delta \Lambda} c} \delta(f(v) - f_0)\, \mathcal{D}v\, \mathcal{D}c'\mathcal{D}c$$

The anticommuting scalars c, c' are the Faddeyev-Popov-Feynman ghost fields. Finally observe that all physical consequences of the theory will not depend on the particular function f_0 which enters the gauge fixing condition. Therefore one can average over it with a suitable weight factor. So, e.g., one can use

$$Z = \int \mathcal{D}f_0\, e^{\frac{i}{2}\int f_0^2} Z_{f_0} = N \int e^{iS(v) + \frac{i}{2}\int f^2 + i\iint c' \frac{\delta f}{\delta \Lambda} c} \mathcal{D}v\, \mathcal{D}c'\mathcal{D}c$$

This last form is convenient for perturbation theory. Both the gauge fixing term and the ghost term appear as additional terms in the action, for which we now have

$$S(v,c,c') = S(v) + \frac{1}{2}\int f^2 d^4x + \iint c'(x) \frac{\delta f}{\delta \Lambda}(x,x') c(x') d^4x\, d^4x'$$

The new action $\hat{S}$ is not gauge invariant, only its term S is. However, it is invariant under the so-called Becchi-Rouet-Stora-Slavnov (BRSS) transformations (see below)

$$\delta v_a = \varepsilon D_a c = \varepsilon (\partial_a c + i[v_a, c])$$
$$\delta c = -i\varepsilon c^2$$
$$\delta c' = -\varepsilon f(v)$$

Here ε is a constant (x-indep.) anticommuting infinitesimal parameter. The ghosts, which are in the adjoint repr., can be described as matrices, just as v_a, but with anticommuting elements. Then $c \propto c_i T_i$

$c^2 = \frac{1}{2}\{c, c\} \propto T_i f_{ijk} c_j c_k$, where f_{ijk} are the structure constants of the group and T_i the matrices representing the generators. Observe that

$$\delta(D_a c) = \delta(\partial_a c + i[v_a, c]) = -i\varepsilon \partial_a c^2 + i[\varepsilon D_a c, c] + [v_a, \varepsilon c^2]$$
$$= -i\varepsilon(\partial_a c\, c + c\, \partial_a c) + i\varepsilon D_a c\, c + i\varepsilon c D_a c + \varepsilon v_a c^2 - \varepsilon c^2 v_a = 0$$

Now

$$\frac{\delta f}{\delta \Lambda} c = \frac{\delta f}{\delta v_a} D_a c$$

so

$$\delta\left(\frac{\delta f}{\delta \Lambda} c\right) = \delta\left(\frac{\delta f}{\delta v_a} D_a c\right) = \frac{\partial f}{\delta v_a \delta v_b} \varepsilon D_b c\, D_a c = 0$$

because the second derivative is symmetric and the ghosts anticommute. This implies

$$\delta\left(\frac{1}{2} f^2 + c' \frac{\delta f}{\delta \Lambda} c\right) = \varepsilon f \frac{\delta f}{\delta v_a} D_a c - \varepsilon f \frac{\delta f}{\delta \Lambda} c = 0$$

On the other hand $\delta S(v) = 0$ (the BRSS transf. is a gauge transf. on v) Finally $\boxed{\delta S(v, c, c') = 0}$

61 GAUGE FIXING IN SUPERSPACE

The susy generalization of $(\partial_a v^a)^2$ is as follows. Let the components of a real superfield V be

$$V = (C, \chi, M, N, v_a, \lambda, D)$$

(real Majorana notation). It is easy to see that

$$A = M\,, \quad B = N, \quad \psi = \lambda + \gamma^i \partial \chi\,, \quad F = \partial_a v^a, \quad G = D + \Box C$$

form a scalar supermultiplet. Its chiral superfield is

$$\Phi \quad \bar{D}^2 V$$

Now remember that $(\Phi \bar{\Phi})_D \propto$ kinetic Lagrangian and that this contains a term $\frac{1}{2} F^2$. This means that

$$(D^2 V \, \bar{D}^2 V)_D \propto (\partial_a v^a) + \cdots$$

This is the susy generalization of the gauge fixing term $(\partial_a v^a)$. The gauge fixing condition is

$$\bar{D}^2 V = 0\,.$$

The gauge transfs have the form (p. 31)

$$e^V \to e^{-i\Lambda^\dagger} e^V e^{i\Lambda} \qquad \bar{D}_{\dot\alpha} \Lambda = D_\alpha \Lambda^\dagger = 0\,.$$

Therefore the ghosts will be, like the gauge parameters, chiral chiral superfields c, c' in the complex extension of the adjoint repn.

$$\bar{D}_{\dot\alpha} c = \bar{D}_{\dot\alpha} c' = D_\alpha c^\dagger = D_\alpha c'^\dagger = 0$$

c and c' are anticommuting.

In order to find the action with gauge fixing and ghost term, we need the infinitesimal form of a gauge transf. on V. For this we develope some useful identities for non commuting quantities.

1) $e^{-X}\delta e^{X} = \int_0^1 e^{-\tau X}\,\delta X\, e^{\tau X}\,d\tau$ for any variation δ.

Proof: let

$$g(\tau) \equiv e^{-\tau X}\delta e^{\tau X} \qquad g(0) = 0$$

$$\frac{dg}{d\tau} = [g, X] + \delta X \qquad \left.\frac{dg}{d\tau}\right|_{\tau=0} = \delta X$$

Define $f(\tau) = \frac{dg}{d\tau}$. It satisfies

$$\frac{df}{d\tau} = [f, X] \;,\; f(0) = \delta X$$

Solution

$$f(\tau) = e^{-\tau X}\,\delta X\, e^{\tau X}$$

Integrate from zero to one, $g(1) = \int_0^1 f(\tau)\,d\tau$.

2) Define $X \wedge Y = L_X Y = [X, Y]$

$$X^2 \wedge Y = L_{X^2} Y = [X,[X, Y]] \quad \text{etc.}$$

Clearly

$$e^{-X}\, y\, e^{X} = e^{-X} \wedge Y \qquad \text{(prove!)}$$

Then $e^{-X}\delta e^{X} = \int_0^1 e^{-\tau X} \wedge \delta X\, d\tau$

$$= \int_0^1 e^{-\tau X}\, d\tau \wedge \delta X = \frac{1 - e^{-X}}{X} \wedge \delta X \,.$$

We now make use of the identities obtained to find the form of an infinitesimal gauge transf. on V. Remember that, for any variation δ and non commuting quantities X, Y

$$e^{-X}\delta e^{X} = \frac{1-e^{-X}}{X}\wedge\delta X\,, \qquad e^{-X}\,Y\,e^{X} = e^{X}\wedge Y\,.$$

We know that

$$e^{V'} = e^{-i\Lambda^{\dagger}}\,e^{V}\,e^{i\Lambda} \Rightarrow e^{V+\delta V} - e^{V} = -i\Lambda^{\dagger}e^{V} + ie^{V}\Lambda$$

$$\frac{1-e^{-V}}{V}\wedge\delta V = -ie^{-V}\wedge\Lambda^{\dagger} + i\Lambda$$

$$\delta V = -\frac{iVe^{-V}}{1-e^{-V}}\wedge\Lambda^{\dagger} + i\frac{V}{1-e^{-V}}\wedge\Lambda$$

$$= \frac{i}{2}V\wedge(\Lambda+\Lambda^{\dagger}) + \frac{i}{2}V\coth\frac{V}{2}\wedge(\Lambda-\Lambda^{\dagger})$$

We have used the fact that, for functions $f(X)$ $g(X)$ definable as formal power series in X,

$$g(X)\wedge[f(X)\wedge Y] = [g(X)f(X)]\wedge Y$$

As a consequence, the equation

$$f(X)\wedge Y = Z$$

can be solved in the form

$$Y = [f(X)]^{-1}\wedge Z$$

In analogy with the treatment of pp. 57–60 We take ghosts superfields which are like the parameter Λ of the gauge transf., therefore chiral

$$\bar{\mathcal{D}}_{\dot\alpha} C = \bar{\mathcal{D}}_{\dot\alpha} C' = 0 \qquad (C \text{ and } C' \text{ anticommute})$$

complex matrices in the adjoint repr. The BRSS Transf.s will be

$$\delta V = \frac{i}{2}\varepsilon\, V \wedge (C + C^\dagger) + \frac{i}{2}\varepsilon V \coth\frac{V}{2} \wedge (C - C^\dagger)$$

$$\delta C = -i\varepsilon\, C^2 \qquad \varepsilon \text{ anticommuting}$$

(see p. 60)

$$\delta C' = -\varepsilon \bar{\mathcal{D}}^2 \mathcal{D}^2 V$$

Since $\mathcal{D}^2 V = 0$ is the gauge fixing condition (notice the extra $\bar{\mathcal{D}}^2$ because C' is chiral)

The ghost Lagr. will be (see p. 59)

$$S_{ghost} \propto i\,\mathrm{tr} \int d^4x\, d^2\theta\, C' \delta(\bar{\mathcal{D}}^2 V) + \text{c.c.}$$

$$\propto \mathrm{tr} \int d^4x\, d^4\theta\, (C' - C'^\dagger) \left[\frac{V}{2} \wedge (C + C^\dagger) + \frac{V}{2} \coth\frac{V}{2} \wedge (C - C^\dagger) \right]$$

and the gauge fixing term

$$S_{g.f.} \propto \mathrm{tr} \int d^4x\, d^4\theta\, (\mathcal{D}^2 V)(\bar{\mathcal{D}}^2 V).$$

(in the above $g = 1$; the coupling constant must be introduced $V \to gV$). The other pieces of the action are described on pp. 31–33.

PROJECTION OPERATORS

$$\Pi_0 = \Pi_{0+} + \Pi_{0-} \quad , \quad \Pi_{0+} = \frac{\bar{D}^2 D^2}{16\,\Box} \quad , \quad \Pi_{0-} = \frac{D^2 \bar{D}^2}{16\,\Box}$$

$$\Pi_{1/2} = -\frac{D^\alpha \bar{D}^2 D_\alpha}{8\,\Box} = -\frac{\bar{D}_{\dot\alpha} D^2 \bar{D}^{\dot\alpha}}{8\,\Box}$$

They satisfy $\Pi_i \Pi_j = \Pi_i \delta_{ij}$ (no sum)

$$\Pi_0 + \Pi_{1/2} = 1$$

Verify using the comm. rels of pp 16–19.
On the vector components v_a of the superfield, Π_0 corresponds to the longitudinal projection $\frac{\partial_a \partial_b v^b}{\Box}$ and $\Pi_{1/2}$ to the transverse projection $\left(\eta_{ab} - \frac{\partial_a \partial_b}{\Box}\right) v^b$. The action for the free superfield V is (up to a factor)

$$\mathrm{tr} \int d^4x \int d^2\theta\; W^\alpha W_\alpha \propto \mathrm{tr} \int d^4x \int d^4\theta\; D^\alpha V\, \bar{D}^2 D_\alpha V$$

$$\propto \mathrm{tr} \int d^4x \int d^4\theta\; V \,\Box\, \Pi_{1/2} V$$

The gauge fixing term is $\propto$ to

$$\mathrm{tr} \int d^4x \int d^4\theta\; D^2 V\, \bar{D}^2 V \propto \mathrm{tr} \int d^4x\, d^4\theta\; V \{D^2 \bar{D}^2\} V$$

$$\propto \mathrm{tr} \int d^4x\, d^4\theta\; V \,\Box\, \Pi_0 V$$

The sum is $\propto$ (ξ is a gauge fixing parameter)

$$\mathrm{tr} \int d^4x\, d^4\theta\; V \,\Box \left(\Pi_{1/2} + \tfrac{1}{\xi} \Pi_0\right) V.$$ For $\xi = 1$

$$= \mathrm{tr} \int d^4x\, d^4\theta\; V \,\Box V$$

Susy Feynman gauge.

In the Feynman gauge the $\underline{V}\underline{V}$ propagator is just $\propto \frac{1}{p^2}\delta^4(\theta_{12})$.

The free part of the ghost action is obtained by expanding S_{ghost} from p. 64. For $V=0$ one finds

$$\propto \text{tr} \int d^4x d^4\theta \, (C' - C'^\dagger)(C - C^\dagger)$$

$$= -\text{tr} \int d^4x \, d^4\theta \left(C'C^\dagger + C'^\dagger C\right)$$

($\int d^4\theta$ of a chiral superfield gives zero). This looks similar to the kinetic Lagr. of p. 25 except that it mixes C and the "antighost" C'. The $\underline{C}\underline{C'^\dagger}$ propagator for ghosts is like the $\underline{\bar{\phi}}\underline{\phi}$ propagator $\propto \frac{1}{p^2}\delta^4(\theta_{12})$. Actually one finds that the sign is $+$ for $\underline{\phi}\underline{\bar{\phi}}$ and $\underline{C}\underline{C'^\dagger}$ and $-$ for $\underline{V}\underline{V}$. The vertices are obtained by expanding the action. It is non polynomial in gV (V has dimension zero). To calculate to any given order one must expand the action up to that order.

Conclusion: (Grisaru, Roček & Siegel).

→ Massless propagators $\pm \frac{1}{p^2}\delta^4(\theta_{12})$ (+ for $\underline{\phi\bar{\phi}}$ and ghosts, − for $\underline{VV}$)

→ Massive scalar propagators

$$\underline{\phi\bar{\phi}} = \frac{1}{p^2+m^2}\delta^4(\theta_{12}) \quad , \quad \underline{\phi\phi} = \frac{m}{p^2(p^2+m^2)}\frac{1}{4}D^2(p,\theta_1)\delta^4(\theta_{12})$$

→ Vertices by expanding action. Extra $\bar{D}^2$ (or D^2) for each chiral (antichiral) superfield, but omitting one $\bar{D}^2$ (or D^2) for $\int d^2\theta \to \int d^4\theta$ and on $\bar{D}^2$ (or D^2) for a chiral (antichiral) external line.

→ (−1) for each ghost loop.

Power counting

In renormalizable theories each vertex has four D's (from D^2, $\bar{D}^2$ of chiral superfields or D^α, $\bar{D}^2 D_\alpha$ of gauge superfield) except for vertices with one external chiral superfield, where two D's are missing. Dimension of $D \sim m^{1/2} \sim p^{1/2}$. $D^4 \sim p^2$. Let there be in the graph: L loops, P propagators, V vertices, E external lines, E_c external chiral lines, C chiral ($\underline{\phi\phi}$ or $\underline{\bar{\phi}\bar{\phi}}$ propagators; they have an additional $\frac{D^2}{p^2} \sim \frac{1}{p}$ beyond the usual $\frac{1}{p^2}$ of all propagators)

Degree of divergence D. Total of $(D^4)^V \sim p^{2V}$ from vertices, p^{-2P} from propagators, p^{-C} from chiral propagators, p^{4L} from d^4p of each loop, p^{-E_c} from missing D^2 for external chiral line. Furthermore each loop uses up a $D^2\bar{D}^2 \sim p^2$ from $\delta D^2\bar{D}^2\delta = \delta$, which gives p^{-2L}. So

$$D = 4L - 2L - 2P + 2V - C - E_c$$

Now use $L - P + V = 1$ (Euler)

$$\boxed{D = 2 - C - E_c}$$

If all masses vanish $\underline{\phi\phi} = 0$ and $C = 0$.

This formula could be obtained more simply by observing that D is the dimension of the effective vertex which, after integration $\int d^4x \int d^4\theta$ gives the action, a pure number. Now, remember that $\int d^4\theta \sim \left(\frac{\partial}{\partial\theta}\right)^4 \sim p^2$, external chiral lines carry dimension p, external V lines carry no dimension. Then, since $\underline{\phi\phi}$ and $\underline{\bar{\phi}\bar{\phi}}$ contribute $m \sim p$,

$$0 = -4 + 2 + E_c + C + D$$

same as above. Compare with ordinary QED. Let E_F and E_B be the numbers of external fermion and boson lines, which carry dimension $\frac{3}{2}$ and 1 respectively. So $0 = -4 + \frac{3}{2}E_F + E_B + D$

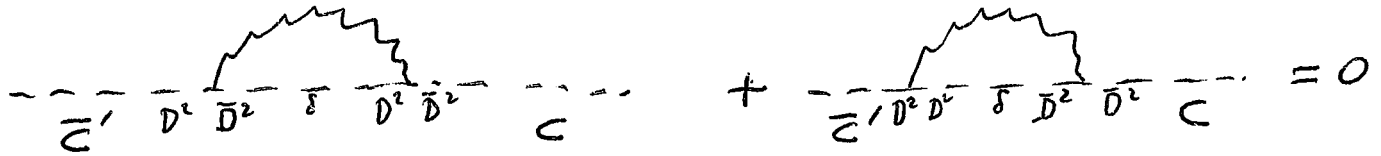
$$D = 4 - \frac{3}{2}E_F - E_B$$

This formula starts with 4, the susy formula with 2.

Example: one loop corrections to the $N=4$ susy Yang-Mills theory, written in terms of $N=1$ superfields as on p. 36. From Grisaru, Roček & Siegel. Theory is finite at one loop in susy Feynman gauge.

<u>Ghost propagator corrections</u>

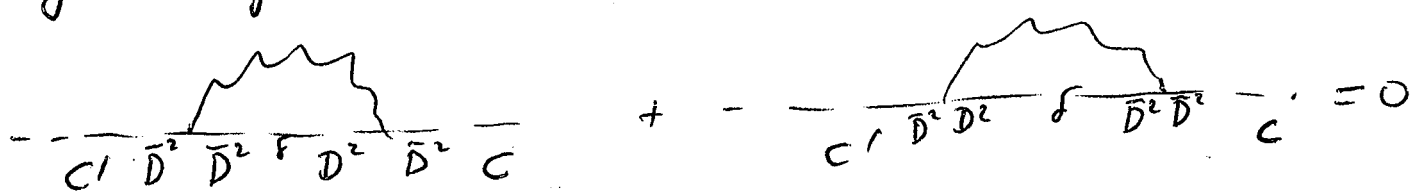
$\sim\sim$ V propagator
$- - -$ ghost propagator



These cancel even in the unextended ($N=1$) theory just due to signs in the interaction. In this case

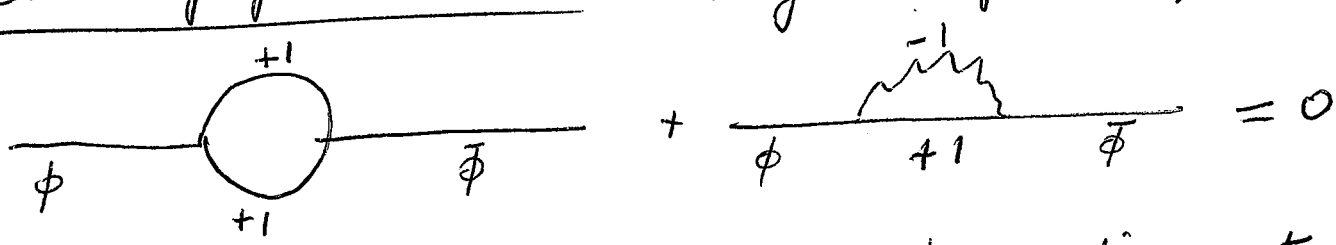
$$D = 2 - E_c \qquad \text{since} \qquad m = 0$$

and for the above diagrams $D = 0$, which means log divergence! Also



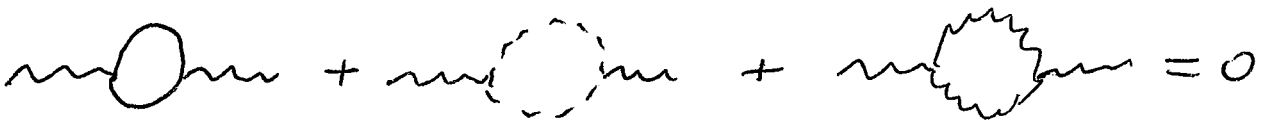
This cancellation is expected because all corrections must be $\int d^4\theta$.

<u>Chiral prop. corrections.</u> Diagrams of the type



These diagrams are ($D = 2 - 2 = 0$) log divergent for the cancellation the number (3) of chiral superfields and the particular trilinear coupling of p. 36 are essential. The opposite sign of the $\underbrace{VV}$ prop. with respect to the $\phi\underbrace{\bar{\phi}}$ prop. does the trick. —— $\phi\underbrace{\bar{\phi}}$ propag.

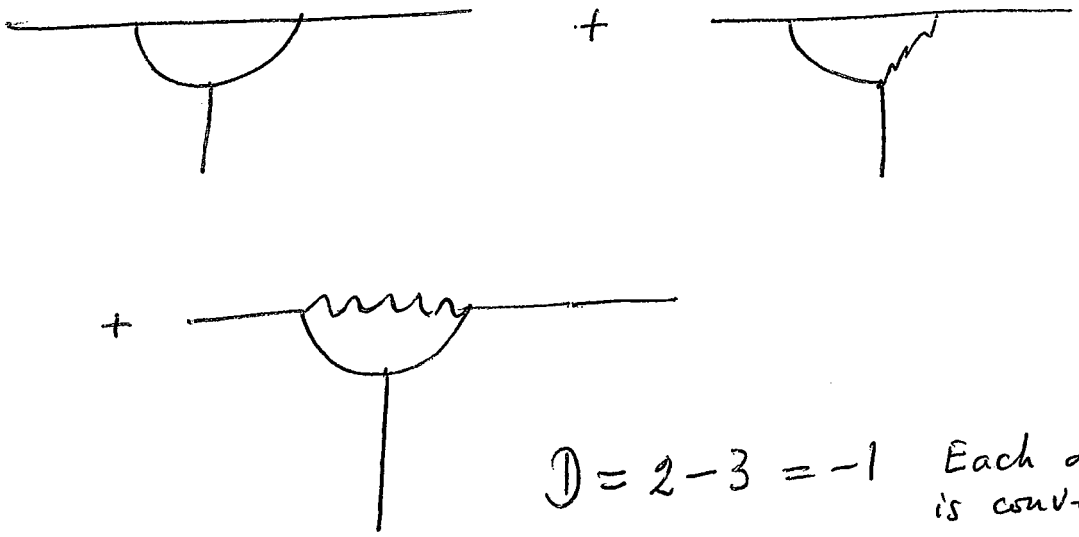
Gauge propagator corrections



$3\Pi_{1/2}$ $\quad -\frac{1}{2}\Pi_{1/2} - \frac{1}{2}\Pi_0$ $\quad -\frac{5}{2}\Pi_{1/2} + \frac{1}{2}\Pi_0$

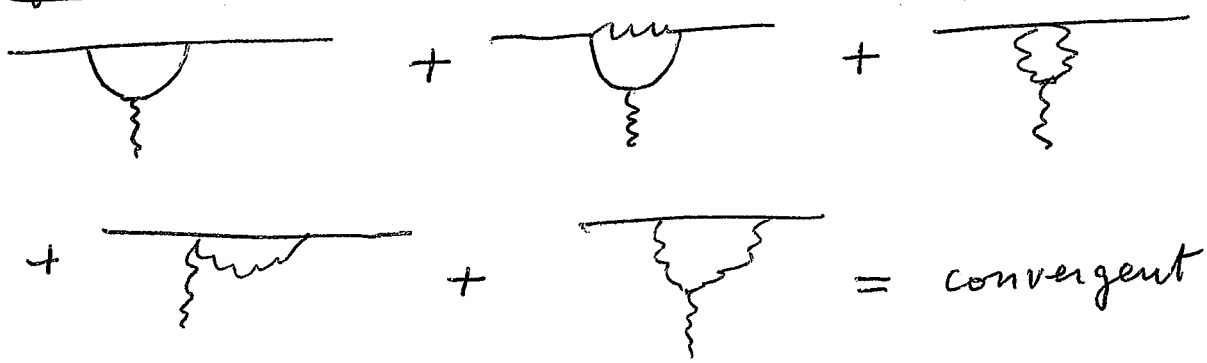
In this cancellation the trilinear coupling of $\beta 36$ does not enter, but the number (3) of chiral superfields does. These diagrams have $D = 2-0 = 2$ but gauge invariance would anyway make the divergence only log.

Chiral vertex corrections



$D = 2-3 = -1$ Each diagram is convergent

Gauge vertex corrections



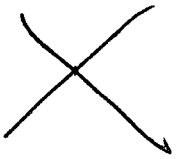
$D = 2-2 = 0$ Each diagram is log div.

71 Let us switch to a different technique

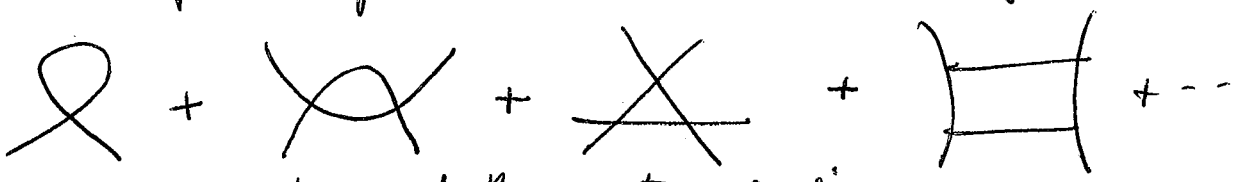
Coleman - Weinberg (Erick) formula for the effective one-loop potential.

Example: $\frac{\lambda}{4!}A^4$ theory. Tree potential

$$V_0 = \frac{\lambda}{4!}A^4$$



One loop one-particle irreducible diagrams



Zero momentum of the external lines.

Sum

$$i\int\frac{d^4p}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{-\frac{\lambda}{2}A^2}{p^2+m^2-i\varepsilon}\right)^n \longrightarrow \text{Wick rotation}$$

$$\rightarrow \frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\ln\left(1+\frac{\frac{\lambda}{2}A^2}{p^2+m^2}\right) \quad \text{divergent integral.}$$

Use $\int d^4p\,\ln(p^2+x) = \frac{\pi^2}{2}x^2\ln x + q(x)$

$$q(x) = \int \ln p^2\, d^4p + x\int\frac{d^4p}{p^2} - \frac{x^2}{2}\int\frac{d^4p}{(p^2)^2}$$

(For instance, take $\frac{\partial}{\partial x}$ and evaluate $\int\frac{d^4p}{p^2+x}$)

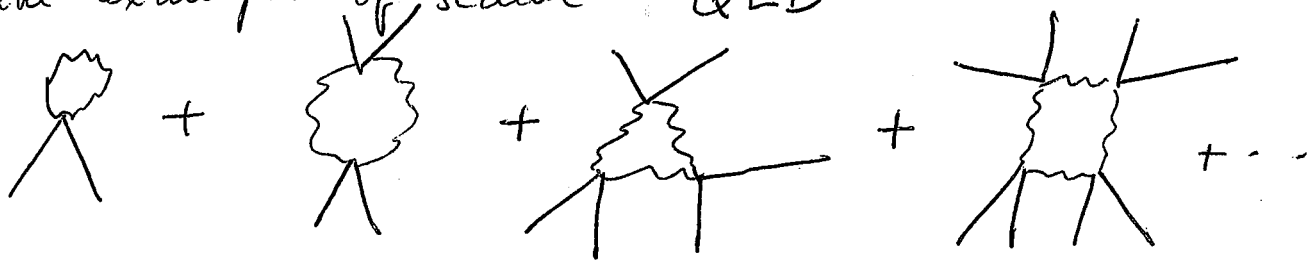
Divergences in $q(x)$ can be absorbed into a shift of the vacuum energy, a mass renorm. and a coupling constant renorm. Conclusion: one loop potential

$$V_1 = \frac{1}{64\pi^2}\left(m^2+\frac{\lambda A^2}{2}\right)^2\ln\frac{m^2+\frac{\lambda}{2}A^2}{\mu^2}\ ;\ \mu \text{ renormaliz. point.}$$

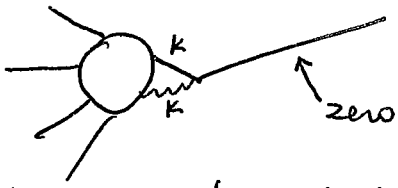
General $V_1 = \frac{1}{64\pi^2}\mathrm{tr}\left[(M^2)^2\ln\frac{M^2}{\mu^2}\right]$; $M^2_{ij} = \frac{\partial^2 V_0}{\partial A_i \partial A_j}$.

Contribution of gauge field loops ~~~

Take example of scalar QED



from sea-gull vertices. Diagrams of the form



vanish in the Landau gauge

The momentum of the scalar is equal to that of the vector, the coupling is $\propto k_a$ and the propagator is transverse $\left(\eta_{ab} - \frac{k_a k_b}{k^2}\right)\frac{1}{k^2}$. Result

$$V_1 = \frac{3e^4}{64\pi^2}|A|^4 \ln\frac{e^2|A|}{\mu^2}$$; e electric charge, the field A is complex.

The factor 3 is from the polarizations of the photon, otherwise the calculation is the same as on p. 71

General form for a gauge theory. Define the gauge field mass matrix from.

$$L = \cdots\cdots - \frac{1}{2} M^2_{ij}(A) v_{ai} v^a_j + \cdots$$

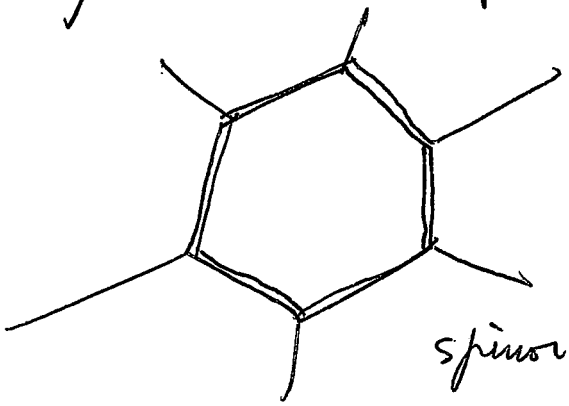
Then

$$V_1 = \frac{3}{64\pi^2}\,\mathrm{tr}\,(M^2)^2 \ln\frac{M^2}{\mu^2}$$

M^2_{ij} will have a form $\sim g^2(T_i A, T_j A)$

where T_i are the matrices representing the generators.

The contribution of spinors is calculated in a similar way as a sum of diagrams of the form



The double lines describe spinor propagators. Spinor loops carry a (−) sign.

One can always split a Dirac spinor ψ into Majorana spinors φ and χ: $\psi = \frac{1}{\sqrt{2}}(\varphi + i\chi)$. For Majorana spinors the mass matrix is defined as $L = \dots - \frac{i}{2}\bar{\varphi}_i \hat{M}_{ij}(A)\varphi_j + \dots$

In general $\hat{M} = M_1 + \gamma_5 M_2$. Consider $M = M_1 + iM_2$.

The eigenvalues of MM^+ are the squares of the masses

One finds

$$V_1 = -\frac{1}{32\pi^2}\,\mathrm{tr}\left[(MM^+)^2 \ln\frac{MM^+}{\mu^2}\right]$$

Finally, the total one-loop effective potential is

$$V_{1\,\mathrm{total}} = \frac{1}{64\pi^2}\,\mathrm{tr}\left[(M^2_{\mathrm{scalar}})^2 \ln\frac{M^2_{\mathrm{scalar}}}{\mu^2}\right] + \frac{3}{64\pi^2}\,\mathrm{tr}\left[(M^2_{\mathrm{vector}})^2 \ln\frac{M^2_{\mathrm{vector}}}{\mu^2}\right]$$
$$- \frac{1}{32\pi^2}\,\mathrm{tr}\left[(MM^+)^2 \ln\frac{MM^+}{\mu^2}\right]$$

The coefficients are such that the sum would cancel for a supermultiplet if all masses are equal (e.g. 2 real scalars and one Maj. spinor; or 1 real vector, 2 Maj spinors and 1 real scalar).

A very tricky point is the range of validity of the one-loop Coleman–Weinberg formula. Often one uses it simply because it is hard to do better. We shall do just this.

The Coleman - Weinberg formula can be used to show that the vacuum degeneracy is removed in a Susy theory with spontaneously broken susy, since it shows explicitly how the one loop effective potential depends on the scalar field which is undetermined by the minimum condition at the tree level. This has been done for various models by Hug, Phys. Rev. 14, 3548 (1976). Let us see how it works for the O'Raifeartaigh model, pp. 43–44. Take case 1) $A_1 = A_2 = 0$, A_0 arbitrary. At the minimum $V = \lambda^2$ for all values of A_0. However the masses depend on A_0. Let us calculate them.
Observe that the transf. $A_0 \to e^{i\alpha} A_0$, $A_2 \to e^{i\alpha} A_2$ $A_1 \to A_1$ changes the superpotential as $f \to e^{i\alpha} f$. Therefore the potential V is unchanged. We can use this to make $\langle A_0 \rangle$ real. We call real expectation value a. At the minimum we have then

$$\frac{\partial^2 f}{\partial A_i \partial A_j} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2ga & m \\ 0 & m & 0 \end{pmatrix} \qquad \frac{\partial f}{\partial A_i} = \begin{pmatrix} \lambda \\ 0 \\ 0 \end{pmatrix}$$

$$\frac{\partial^3 f}{\partial A_i \partial A_j \partial A_0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2g & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \frac{\partial^3 f}{\partial A_i \partial A_j \partial A_1} = \begin{pmatrix} 0 & 2g & 0 \\ 2g & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\frac{\partial^3 f}{\partial A_i \partial A_j \partial A_2} = 0$$

Therefore, at the minimum

$$\frac{\partial^2 V}{\partial A_i \partial A_j} = \overline{\frac{\partial f}{\partial A_k}} \; \overline{\frac{\partial^3 f}{\partial A_k \partial A_i \partial A_j}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2g\lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\frac{\partial^2 V}{\partial A_i \partial \bar{A}_j} = \overline{\frac{\partial^2 f}{\partial A_i \partial A_k}} \; \overline{\frac{\partial^2 f}{\partial A_k \partial A_j}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & m^2+4g^2a^2 & 2gma \\ 0 & 2gma & m^2 \end{pmatrix}$$

The Fermion square masses are the eigenvalues of this last matrix. They are

$$0 \;,\; m^2+2g^2a^2 \pm 2ga\sqrt{m^2+g^2a^2}$$

The Boson square masses are obtained from

$$\frac{1}{2} A_i \frac{\partial^2 V}{\partial A_i \partial A_j}\bigg|_{\min} A_j + \frac{1}{2} \bar{A}_i \frac{\partial^2 V}{\partial \bar{A}_i \partial \bar{A}_j}\bigg|_{\min} \bar{A}_j + A_i \frac{\partial^2 V}{\partial A_i \partial \bar{A}_j}\bigg|_{\min} \bar{A}_j =$$

$$= g\lambda\left(A_1^2 + \bar{A}_1^2\right) + \left(m^2+4g^2a^2\right) A_1 \bar{A}_1 + m^2 A_2 \bar{A}_2$$

$$+ 2gma\, A_1 \bar{A}_2 + 2gma\, A_2 \bar{A}_1$$

Clearly the complex field A_0 has vanishing mass (two real scalar fields of zero mass) since it does not enter in the mass term. For the rest one must find the eigenvalues of a 4×4 matrix. They turn out to be

$$m^2 + \varepsilon g\lambda + 2g^2a^2 + \eta\sqrt{(2g^2a^2+\varepsilon g\lambda)^2 + 4g^2m^2a^2}$$

where $\varepsilon = \pm 1$, $\eta = \pm 1$. Observe that the sum of the Boson (masses)^2 is $4(m^2+2g^2a^2)$, that of the Fermion (masses)^2 is $2(m^2+2g^2a^2)$, as it should be (the sum rule requires a factor 2 for Majorana Fermions).

Having the eigenvalues of the mass matrices one can use the Coleman – Weinberg formula of p 73 The traces reduce to sums (in our case there are no vectors) and one can calculate the minimum of the effective potential. An interesting simplification occurs. Observe that, in general, (for a chiral model) the Boson (mass)^2 matrix is given by

$$\begin{pmatrix} m m^+ & n \\ n^+ & m^+ m \end{pmatrix} \quad \text{where} \quad m_{ij} = \frac{\partial^2 f}{\partial A_i \partial A_j}$$

is the Fermion mass matrix, and

$$n_{ij} = \frac{\partial f}{\partial A_u} \frac{\partial^3 f}{\partial A_u \partial A_i \partial A_j} \qquad \text{(both } m \text{ and } n \text{ are symmetric)}$$

Taking the trace of the Boson matrix one finds 2 x the trace of the Fermion square mass matrix, a general proof that the sum rule

$$\sum_i w_i m_i^2(A) = 0$$

is satisfied even for the field dependent masses $m_i(A)$. Here w_i are the weights for the various spins (+1 for a real scalar, +3 for a real vector, −2 for a Majorana spinor) which satisfy the supersymmetry rule

$$\sum_i w_i = 0$$

for any supermultiplet. Now we claim that (would vanish if Susy is good)

$$\sum_i w_i m_i^4(A) = \text{constant} > 0$$

i.e. field independent (M. K. Gaillard[76]). To see this,

observe that, for a renormalizable theory, $f(A)$ is a cubic polynomial, and $\frac{\partial^3 f}{\partial A_i \partial A_j \partial A_u}$ is a set of constants. Furthermore, along the valley of minima $F_u = -\frac{\partial f}{\partial A_u}$ is also constant. So the matrix N_{ij} is independent of the scalar fields. For the Bosons the matrix of the (masses)^4 has diagonal terms given by

$$\begin{pmatrix} m m^\dagger m m^\dagger + n n^\dagger & \cdots \\ \cdots & m^\dagger m m^\dagger m + n^\dagger n \end{pmatrix}$$

and that for the Fermions it is simply $m m^\dagger m m^\dagger$.

We have

$$\text{tr (mass Bosons)}^4 - 2\,\text{tr (mass Fermions)}^4 = 2\,\text{tr}\, n n^\dagger$$

which proves our assertion.

Now

$$V_1(a) = \frac{1}{64\pi^2} \sum_i w_i m_i^4 \ln \frac{m_i^2}{\mu^2}$$

$$= \frac{1}{64\pi^2} \left(\sum_i w_i m_i^4 \ln m_i^2 - \sum_i w_i m_i^4 \ln \mu^2 \right)$$

We have shown that the last terms is independent of the scalar fields. Therefore

$$\frac{d}{da} V_1(a) = \frac{1}{64\pi^2} \sum_i w_i \left(\frac{dm_i^4}{da} \ln m_i^2 + \frac{1}{2} \frac{dm_i^4}{da} \right)$$

$$= \frac{1}{64\pi^2} \sum_i w_i \frac{dm_i^4}{da} \ln m_i^2$$

since again, the last term vanishes. These simplifications

are general. We can now plug in the expressions for the masses from p. 75 and calculate $\frac{dV_1(a)}{da}$. We leave this as an exercise for the reader. One finds that $\frac{dV_1}{da}$ vanishes for $a=0$ and that $\frac{d^2V_1}{da^2}$ is positive for $a=0$. Furthermore, one can see that $a=0$ is not only a local minimum but also a global minimum (Hurra!). Therefore, if one is willing to use the one-loop effective potential, the vacuum degeneracy is removed and the value of $\langle\phi_0\rangle$ can be calculated. In any case, it is clear that higher loops do not change the fact that the degeneracy is removed.