PHY 250 (P. Horava) Homework Assignment 1 Solutions
Grader: Uday Varadarajan

1. Problem 1.4 of Polchinski, Vol. 1:

   (a) Consider the states of the open string spectrum at level $N = 2$, all of which have masses given by $m^2 = (N-1)/\alpha' = 1/\alpha'$,
   $$\alpha^i_{-1}\alpha^j_{-1}|0\rangle, \qquad \alpha^i_{-2}|0\rangle, \tag{1.1}$$
   where $i, j = 1, \ldots, D-2$. Note that due to Bose symmetry, the first set of states makes up a *symmetric* 2-tensor of $SO(D-2)$, which decomposes into a symmetric traceless 2-tensor and a scalar of $SO(D-2)$. The second state is just a vector of $SO(D-2)$. Now, a traceless symmetric 2-tensor $e^{IJ} = e^{JI}$, $e^I_I = 0$, $I, J = 1, \ldots, D-1$ of $SO(D-1)$ transforms under an $SO(D-2)$ subgroup as a traceless symmetric 2-tensor $e^{ij} = e^{ji}$, a vector $e^{i(D-1)} = e^{(D-1)i}$, and a scalar $e^{(D-1)(D-1)}$. Thus, the states at level $N = 2$ of the open string sit nicely in the traceless symmetric 2-tensor representation of $SO(D-1)$ as was required.
   For $N = 3$, we have the states,
   $$\alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1}|0\rangle, \quad \alpha^i_{-2}\alpha^j_{-1}|0\rangle, \quad \alpha^i_{-3}|0\rangle, \tag{1.2}$$
   Again, by Bose symmetry, the first set of states is a traceless symmetric 3-tensor and a single vector trace of $SO(D-2)$. However, the second set of states now includes an antisymmetric part, and so consists of a traceless symmetric 2-tensor, an antisymmetric 2-tensor, and a scalar of $SO(D-2)$, while the third set corresponds to a vector of $SO(D-2)$. Now, the antisymmetric 2-tensor $b^{IJ} = -b^{JI}$of $SO(D-1)$ decomposes to an antisymmetric 2-tensor $b^{ij} = -b^{ji}$ and vector $b^{i(D-1)} = -b^{(D-1)i}$ of $SO(D-2)$, while the traceless symmetric 3-tensor $e^{IJK} = e^{JIK} = \cdots$ of $SO(D-1)$ decomposes into a traceless symmetric 3-tensor e^{ijk}, a traceless symmetric 2-tensor $e^{ij(D-1)} = e^{i(D-1)j} = e^{(D-1)j} = \cdots$, a vector $e^{i(D-1)(D-1)} = e^{(D-1)i(D-1)} = e^{(D-1)(D-1)i}$, and a scalar $e^{(D-1)(D-1)(D-1)}$ of $SO(D-2)$. Thus, we find that at level $N = 3$, the states of an open string combine to form an antisymmetric 2-tensor and a symmetric traceless 3-tensor of $SO(D-1)$.

   (b) Note that the closed string at some level $N = \tilde{N}$ is just the tensor product of two copies of the open string at level N, so we find that the closed string at level $N = 2$ just consists of a tensor product of two traceless symmetric 2-tensors $e^{IJ}\tilde{e}^{KL} = (I \leftrightarrow J, K \leftrightarrow L)$, of $SO(D-1)$.

2. We consider the twisted sector of an orientifold of closed oriented bosonic strings in flat $\mathbb{R}^{26}$. That is, we impose the conditions that
   $$X^\mu(\tau, \sigma + \ell) = X^\mu(\tau, \ell - \sigma) \tag{1.3}$$
   $$X^\mu(\tau, \sigma - \ell) = X^\mu(\tau, \ell - \sigma). \tag{1.4}$$
   We will work in light-cone gauge and look for a general solution to these boundary conditions. Note that the combination of the two boundary conditions requires that $X^\mu(\tau, \sigma+\ell) = X^\mu(\tau, \sigma - \ell)$ is periodic in σ with period 2ℓ. Thus, we start with the expansion (ignoring numerical factors),
   $$X^i(\tau, \sigma) = x^i + \frac{p^i}{p^+}\tau + C\sum_{n\neq 0}\left\{\frac{\alpha^i_n}{n}e^{-\frac{\pi i n}{\ell}(\sigma + c\tau)} + \frac{\tilde{\alpha}^i_n}{n}e^{-\frac{\pi i n}{\ell}(\sigma - c\tau)}\right\}. \tag{1.5}$$
   Now, we impose the first condition, which we can think of as a $\mathbb{Z}_2$ orbifold of the *worldsheet*, with two fixed points. This condition is satisfied by requiring that $\tilde{\alpha}^i_n = -\alpha^i_{-n}$. Note that as a result, the second condition is automatically satisfied, and we are left with a single set of independent oscillators, just as in the case of the open string,
   $$\begin{aligned} X^i(\tau, \sigma) &= x^i + \frac{p^i}{p^+}\tau + C\sum_{n\neq 0}\left\{\frac{\alpha^i_n}{n}e^{-\frac{\pi i n}{\ell}(\sigma + c\tau)} + \frac{\alpha^i_n}{n}e^{\frac{\pi i n}{\ell}(\sigma - c\tau)}\right\} \\ &= x^i + \frac{p^i}{p^+}\tau + C\sum_{n\neq 0}\frac{\alpha^i_n}{n}e^{-\frac{\pi i n c\tau}{\ell}}\cos\frac{\pi i n\sigma}{\ell}. \end{aligned} \tag{1.6}$$

In fact, we can interpret this twisted sector as unoriented open strings corresponding to fluctuations of a space-filling D-brane. With this interpretation, we see that both conditions above are needed to ensure that a pair of boundaries (the two fixed points) appear a finite length apart on the worldsheet with Neumann-like boundary conditions.

3. Problem 1.9 of Polchinski, Vol 1: We consider closed oriented bosonic strings on $\mathbb{R}^{26}/\mathbb{Z}_2$, where the orbifold acts by reflection in the X^{25} direction. The oscillator expansion is the same as in the unwrapped closed string for X^i, $i = 2, \dots, 24$,

$$X^i(\tau,\sigma) = x^i + \frac{p^i}{p^+}\tau + i\left(\frac{\alpha'}{2}\right)^{\frac{1}{2}} \sum_{n\neq 0} \left\{ \frac{\alpha_n^i}{n} e^{-\frac{2\pi i n}{\ell}(\sigma + c\tau)} + \frac{\tilde{\alpha}_n^i}{n} e^{-\frac{2\pi i n}{\ell}(\sigma - c\tau)} \right\}. \tag{1.7}$$

However, X^{25} for a twisted sector state must be *antiperiodic*, which eliminates the constant modes and requires that the oscillators be half-integrally moded,

$$X^{25}(\tau,\sigma) = i\left(\frac{\alpha'}{2}\right)^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} \left\{ \frac{\alpha^{25}_{n+\frac{1}{2}}}{n+\frac{1}{2}} e^{-\frac{2\pi i (n+\frac{1}{2})}{\ell}(\sigma + c\tau)} + \frac{\tilde{\alpha}^{25}_{n+\frac{1}{2}}}{n+\frac{1}{2}} e^{-\frac{2\pi i (n+\frac{1}{2})}{\ell}(\sigma - c\tau)} \right\}. \tag{1.8}$$

First note that the lack of zero modes corresponding to position and momentum in the x^{25} direction implies that the twisted sector states are localized to the origin in x^{25}. Second, note that reality requires that

$$(\alpha^{25}_{n+\frac{1}{2}})^\dagger = \alpha^{25}_{-n-\frac{1}{2}} \quad (\tilde{\alpha}^{25}_{n+\frac{1}{2}})^\dagger = \tilde{\alpha}^{25}_{-n-\frac{1}{2}} \tag{1.9}$$

From this expression we can guess that the appropriate commutation relation for the oscillators must be

$$[\alpha^{25}_{n+\frac{1}{2}}, \alpha^{25}_{-m-\frac{1}{2}}] = (n+\frac{1}{2})\delta_{nm} \quad [\tilde{\alpha}^{25}_{n+\frac{1}{2}}, \tilde{\alpha}^{25}_{-m-\frac{1}{2}}] = (n+\frac{1}{2})\delta_{nm}. \tag{1.10}$$

More precisely, these commutation relations are precisely what are needed to reproduce the canonical equal time commutation relations

$$[\Pi^{25}(\sigma), X^{25}(\sigma')] = \delta(\sigma - \sigma'), \tag{1.11}$$

where $\Pi^{25} = \frac{p^+}{\ell}\partial_\tau X^{25}$ is the momentum conjugate to X^{25} in light-cone gauge. Plugging the oscillator expansions into the Hamiltonian in light-cone gauge given by,

$$\begin{aligned}
H =& \frac{\ell}{4\pi\alpha' p^+}\int_0^\ell d\sigma \left(2\pi\alpha'\Pi^i\Pi^i + \frac{1}{2\pi\alpha'}\partial_\sigma X^i \partial_\sigma X^i\right) \\
=& \frac{p^i p^i}{2p^+} + \frac{1}{2\alpha' p^+}\sum_{n\neq 0, i\neq 25} (\alpha^i_n \alpha^i_{-n} + \tilde{\alpha}^i_n\tilde{\alpha}^i_{-n}) + \frac{1}{2\alpha' p^+}\sum_{n=-\infty}^{\infty}\left(\alpha^{25}_{n+\frac{1}{2}}\alpha^{25}_{-n-\frac{1}{2}} + \tilde{\alpha}^{25}_{n+\frac{1}{2}}\tilde{\alpha}^{25}_{-n-\frac{1}{2}}\right) \\
=& \frac{p^i p^i}{2p^+} + \frac{1}{\alpha' p^+}\sum_{n=1}^{\infty}\left(\alpha^i_{-n}\alpha^i_n + \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n + \alpha^{25}_{-n+\frac{1}{2}}\alpha^{25}_{n-\frac{1}{2}} + \tilde{\alpha}^{25}_{-n+\frac{1}{2}}\tilde{\alpha}^{25}_{n-\frac{1}{2}} + (D-3)n + (n-1/2)\right) \\
=& \frac{p^i p^i}{2p^+} + \frac{1}{\alpha' p^+}\sum_{n=1}^{\infty}\left[N + \tilde{N} - \frac{D-3}{12} + \frac{1}{24} - \frac{1}{8}(2(\frac{1}{2}) - 1)^2\right].
\end{aligned} \tag{1.12}$$

We have used, in the last line, the heuristic result from Polchinski Problem 1.5 (eq. 2.9.19 of Polchinski Vol. 1)

$$\sum_{n=1}^{\infty}(n - \theta) = \frac{1}{24} - \frac{1}{8}(2\theta - 1)^2, \tag{1.13}$$

to evaluate the ordering constants. Note that the number operators are generally half-integral, due to the half-integral moding of X^{25}. This gives rise to the massive spectrum,

$$m^2 = 2p^+H - p^i p^i = \frac{2}{\alpha'}\left[N + \tilde{N} - \frac{15}{8}\right] \tag{1.14}$$

Translational invariance on the worldsheet imposes the condition that

$$P = -\int_0^\ell d\sigma \Pi^i \partial_\sigma X^i = \frac{2\pi}{\ell}(N - \tilde{N}) = 0. \tag{1.15}$$

Thus, we find that the specrum is

$$m^2 = \frac{4}{\alpha'}\left[N - \frac{15}{16}\right], \tag{1.16}$$

where $N = \tilde{N}$ can be half-integral.