Quadratic shift symmetry

Consider the following nonrelativistic scalar field theory (known as the "Lifshitz scalar") in $D+1$ dimensions (which we will parametrize by Cartesian coordinates t and x^i, $i = 1, \ldots, D$):

$$S = \frac{1}{2} \int dt d^D x \left\{ (\partial_t \phi)^2 - (\partial_i \partial_i \phi)^2 \right\}.$$

This theory is invariant under the following "quadratic shift" symmetry,

$$\phi \to \phi + a_{ij} x^i x^j + a_i x^i + a,$$

where a_{ij}, a_i and a are real, spacetime-independent constants.

Derive the Noether current for this symmetry (following the strategy outlined in Chapter 2.2 of [PS]), and prove that it is conserved.

Reference: This symmetry has played a central role in our paper from last year, on *Multicritical Symmetry Breaking and Naturalness of Slow Nambu-Goldstone Bosons*, Phys. Rev. **D88** (2013) 101701 [arXiv:1308.5967] (by T. Griffin, K. Grosvenor, Z. Yan & P.H.).