Lifshitz Gravity and Lifshitz Holography

❀ Petr Hořava ❀

Berkeley Center for Theoretical Physics

lecture in the Physics 234B course

BCTP, UC Berkeley

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Outline

I. Review of quantum gravity with anisotropic scaling

anisotropic Weyl symmetries & anomalies

detailed balance

II. Lifshitz holography

Lifshitz spacetimes

anisotropic conformal infinity

IIa. Lifshitz holography in relativistic gravity

holographic renormalization, counterterms

anisotropic Weyl anomalies in detailed balance

detailed balance from recursions among holographic counterterms

IIb. Lifshitz holography in Lifshitz gravity

Lifshitz spacetime as a vacuum solution

general form of anisotropic Weyl anomalies with examples

general holographic Weyl anomalies from Lifshitz gravity

III. Black holes & Conclusions

Some references

initial papers on gravity with anisotropic scaling:

. . ., arXiv:0812.4287, arXiv:0901.3775, . . .

brief recent(ish) review:

arXiv:1101.1081 (GR 19 Proceedings).

focus of this talk:

work with Tom Griffin and Charles Melby-Thompson
in arXiv:1112.5660, 1211.4872, 1301.nnnn

work with Tom Griffin and Omid Saremi
in arXiv:to appear very soon

I.
Review: Gravity with anisotropic scaling

Gravity with anisotropic scaling

(also known as Hořava-Lifshitz gravity)

Field theory with anisotropic scaling ($\mathbf{x} = \{x^i, i = 1, \ldots D\}$):

$$\mathbf{x} \to \lambda \mathbf{x}, \quad t \to \lambda^z t.$$

z: dynamical critical exponent – characteristic of RG fixed point.

Many interesting examples: $z = 1$, 2, ..., n, ...
fractions: 3/2 (KPZ surface growth in $D = 1$), ..., $1/n$, ...
families with z varying continuously ...

Condensed matter, dynamical critical phenomena, quantum critical systems, ...

Goal: Extend to gravity, with propagating gravitons, formulated as a quantum field theory of the metric.

Example: Lifshitz scalar [Lifshitz, 1941]

Gaussian fixed point with $z = 2$ anisotropic scaling:

$$S = S_K - S_V = \frac{1}{2} \int dt\, d^D\mathbf{x} \left\{ \dot{\Phi}^2 - (\Delta\Phi)^2 \right\},$$

(Δ is the spatial Laplacian).

Compare with the Euclidean field theory

$$W = -\frac{1}{2} \int d^d x\, (\partial\phi)^2.$$

Shift in the (lower) critical dimension:

$$[\phi] = \frac{d-2}{2}, \quad [\Phi] = \frac{D-2}{2}.$$

Gravity at a Lifshitz point

Spacetime structure: Preferred foliation by leaves of constant time (avoids the "problem of time").

Fields: Start with the spacetime metric in ADM decomposition: the spatial metric g_{ij}, the lapse function N, the shift vector N_i.

Symmetries: foliation-preserving diffeomorphisms, $\mathrm{Diff}(M, \mathcal{F})$.

Action: $S = S_K - S_V$, with

$$S_K = \frac{1}{\kappa^2} \int dt\, d^D\mathbf{x} \sqrt{g} N \left(K_{ij} K^{ij} - \lambda K^2 \right)$$

where $K_{ij} = \dfrac{1}{N} \left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i \right)$ the extrinsic curvature,

$$\text{and} \quad S_V = \frac{1}{\kappa_V^2} \int dt\, d^D\mathbf{x} \sqrt{g} N\, \mathcal{V}(R_{ijk\ell}, \nabla_i).$$

Projectable and nonprojectable theory

N, N_i are the gauge fields for the $\mathrm{Diff}(M, \mathcal{F})$ symmetries generated by $\delta t = f(t)$, $\delta x^i = \xi^i(t, \mathbf{x})$. Hence:

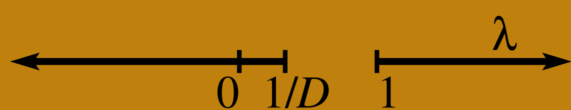
(1) we can restrict $N(t)$ to be a function of time only: projectable theory.

(2) or, we allow $N(t, \mathbf{x})$ to be a spacetime field. New terms, containing $\nabla_i N/N$, are then allowed in S by symmetries: nonprojectable theory.

Spectrum: Tensor graviton polarizations, plus an extra scalar graviton. Three options for the scalar: Live with it, gap it, or eliminate it by an extended gauge symmetry.

Dispersion relation: Nonrelativistic, $\omega^2 \sim k^{2z}$, around this Gaussian fixed point.

Allowed range of λ:



Special case: Detailed balance

Sometimes, the potential term happens to be the square of EoM for some local action W in D dimensions:
This gives theories in detailed balance.
(Terminology borrowed from non-equilibrium stat-mech.)

Example: Lifshitz scalar satisfies the detailed balance condition,

$$(\Delta\Phi)^2 = \left(\frac{\delta W}{\delta\Phi}\right)^2, \quad \text{with } W = -\frac{1}{2}\int d^D\mathbf{x}\,(\partial\Phi)^2.$$

Simplest examples involving (projectable) gravity:

$D = 2: \quad W = \int \sqrt{g}R \quad \to \quad \mathcal{V} = 0,$

$D = 3: \quad W = \int \omega_3(\Gamma) \quad \to \quad \mathcal{V} = C_{ij}C^{ij}.$
(here $\omega_3(\Gamma) = \Gamma \wedge d\Gamma + \frac{2}{3}\Gamma \wedge \Gamma \wedge \Gamma$, and C_{ij} is the Cotton tensor.)

RG flows

Assume $z > 1$ UV fixed point. Relevant deformations trigger RG flow to lower values of z. Example: Lifshitz scalar.

$$S = \frac{1}{2}\int dt\, d^D\mathbf{x} \left\{ \dot{\Phi}^2 - (\Delta\Phi)^2 - \mu^2 \partial_i\Phi\partial_i\Phi - m^4\Phi^2 \right\},$$

Multicriticality. New phases: modulated.

Similarly for gravity:

$$S = \frac{1}{\kappa^2}\int dt\, d^D\mathbf{x}\sqrt{g}N \left\{ K_{ij}K^{ij} - \lambda K^2 - \ldots - \mu^{2z-2}R - M^{2z} \right\}.$$

Flows in IR to $z = 1$ scaling. In the IR regime, S_V is dominated by the spatial part of Einstein-Hilbert.

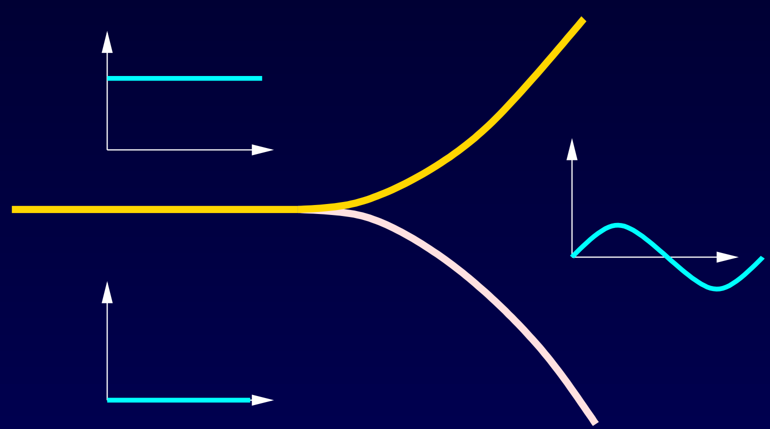
(The $z > 1$ Gaussian gravity fixed points also emerge in IR in condensed matter lattice models, [Cenke Xu & PH].)

Relevant deformations, RG flows, phases

The Lifshitz scalar can be deformed by relevant terms:

$$S = \frac{1}{2}\int dt\, d^D\mathbf{x}\,\left\{\dot\phi^2 - (\Delta\phi)^2 - \mu^2\partial_i\phi\partial_i\phi + m^4\phi^2 - \phi^4\right\}$$

The undeformed $z = 2$ theory describes a tricritical point, connecting three phases – disordered, ordered, spatially modulated ("striped") [A. Michelson, 1976]:

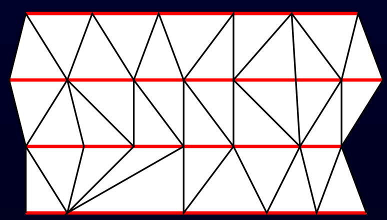


Gravity on the lattice

Causal dynamical triangulations approach [Ambjørn,Jurkiewicz,Loll] to $3+1$ lattice gravity:

Naive sum over triangulations does not work (branched polymers, crumpled phases).

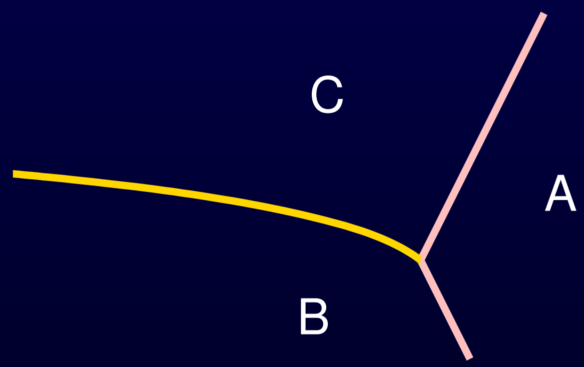
Modify the rules, include a preferred causal structure:



With this relevant change of the rules, a continuum limit appears to exist: The spectral dimension $d_s \approx 4$ in IR, and $d_s \approx 2$ in UV. Continuum gravity with anisotropic scaling: $d_s = 1 + D/z$. ([Benedetti,Henson,2009]: works in $2+1$ as well.)

Phase structure in the CDT approach

Compare the phase diagram in the causal dynamical triangulations:
[Ambjørn et al, 1002.3298]



Note: $z = 2$ is sufficient to explain three phases.
Possibility of a nontrivial $z \approx 2$ fixed point in $3 + 1$ dimensions?

RG flows in gravity: $z = 1$ in IR

Theories with $z > 1$ represent candidates for the UV description. Under relevant deformations, the theory will flow in the IR. Relevant terms in the potential:

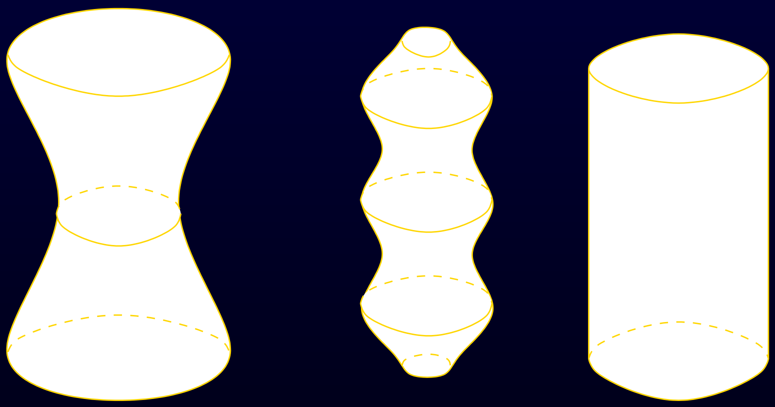
$$\Delta S_V = \int dt\, d^D\mathbf{x} \sqrt{g} N \left\{ \ldots + \mu^2 (R - 2\Lambda) \right\}.$$

the dispersion relation changes in IR to $\omega^2 \sim k^2 + \ldots$
the IR speed of light is given by a combination of the couplings
μ^2 combines with $\kappa, \ldots$ to give an effective G_N.

Sign of k^2 in dispersion relation is opposite for the scalar and the tensor modes! Can we classify the phases of gravity? Can gravity be in a modulated phase?

Spatially homogeneous isotropic phases of gravity

vglue-.5in Examples of phases of gravity with $k = 1$: a de Sitter-like phase, an oscillating cosmology (="temporally modulated" phase); the Einstein static universe appears at the phase transition line, where the theory satisfies detailed balance.

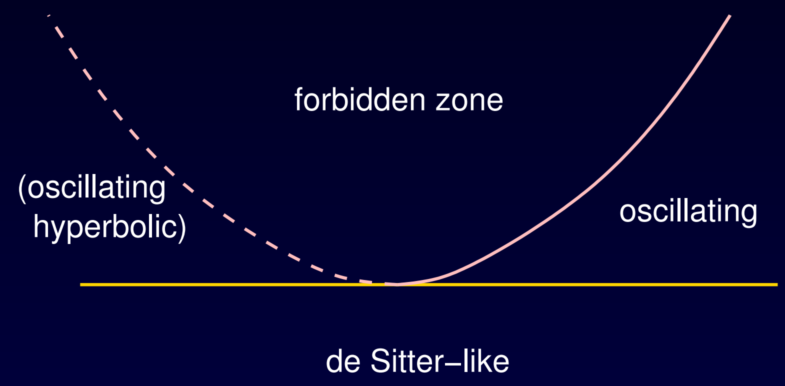


Cosmology: [Kiritsis et al, Brandenberger et al, Lüst et al, many others]

Modulated phases of gravity

[w/ Kevin Grosvenor and Charles Melby-Thompson]

First, classify all spatially homogeneous and isotropic phases. Take $g_{ij} = a^2(t)\gamma_{ij}(k)$, with $k = 0, \pm 1$; set $N_i = 0$. The phase diagram for $k = 1$ (at fixed R^2 terms):



Governed by $\int \mathcal{H}_\perp \equiv (\dot{g})^2 + R^2 + \mu^2 R - 2\Lambda = 0$, the Friedmann equation.

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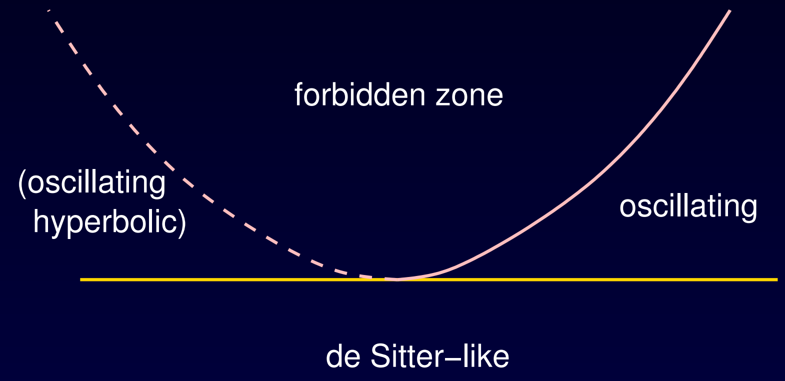


Governed by $\int \mathcal{H}_\perp \equiv (\dot{g})^2 + R^2 + \mu^2 R - 2\Lambda = 0$, the Friedmann equation.

cf.: CDT phase structure in $2+1$ dimensions [arXiv:1111.6634, w/ C. Anderson, S. Carlip, J.H. Cooperman, R. Kommu and P. Zulkowski.]

Deconfinement of $\int \mathcal{H}_\perp$?

Anisotropic Weyl symmetries

Using a spacetime-dependent scaling factor $\Omega(t, \mathbf{x}) = e^{\omega(t,\mathbf{x})}$, define

$$g_{ij} \to \Omega^2 g_{ij}, \quad N_i \to \Omega^2 N_i, \quad N \to \Omega^z N.$$

Such anisotropic Weyl transformations are compatible with foliation-preserving diffeos:

$$\mathrm{Weyl}_z(M, \mathcal{F}) \otimes \mathrm{Diff}(M, \mathcal{F}).$$

Indeed, $[\delta_\omega, \delta_{f,\xi^i}] = \delta_{f\dot{\omega}+\xi^i\partial_i\omega}$.

This provides the appropriate generalization of local "conformal transformations" to foliated spacetimes with anisotropic scaling.

Often, it is sufficient to have the preferred foliation and the anisotropic Weyl transformations defined only asymptotically, "near infinity."

Anisotropic Weyl anomalies

Consider an “anisotropic CFT” in $D+1$ dimensions, with dynamical exponent z.

Place it in a gravitational background. The theory may exhibit anisotropic Weyl anomaly:

$$\delta_\omega W_{\rm eff} = \int dt\, d^D\mathbf{x}\sqrt{g}N\,\omega(t,\mathbf{x})\mathcal{A}(t,\mathbf{x}).$$

What terms can appear in $\mathcal{A}$? Question in BRST cohomology of $\mathrm{Weyl}_z(M,\mathcal{F})$.

Answer for $D=2$, $z=2$: Assuming time reversal invariance, two independent anomaly terms (hence two “central charges”)

$$\mathcal{A} = c_K\left(K_{ij}K^{ij} - \frac{1}{2}K^2\right) + c_V\left[R - \left(\frac{\nabla N}{N}\right)^2 + \frac{\Delta N}{N}\right]^2.$$

Conformal gravity at a Lifshitz point

For some fixed z, extend the gauge symmetry from $\mathrm{Diff}(M,\mathcal{F})$ to $\mathrm{Weyl}_z(M,\mathcal{F}) \otimes \mathrm{Diff}(M,\mathcal{F})$.

Then:

- The kinetic term S_K will be invariant if $z = D$ and $\lambda = 1/D$.
- The theory is automatically nonprojectable.
- The classical theory may be invariant, but the quantum theory generally expected to develop a Weyl anomaly.
- Choice: Theory may or may not satisfy detailed balance.

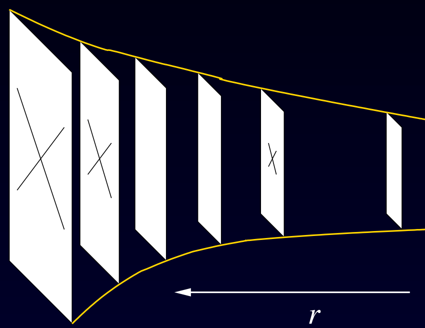
Example: Conformal $z = 2$ gravity in $D = 2$ with detailed balance,

$$S = S_K = \frac{1}{\kappa^2} \int dt\, d^2\mathbf{x} \sqrt{g} N \left(K_{ij} K^{ij} - \frac{1}{2} K^2 \right).$$

II.
Lifshitz holography

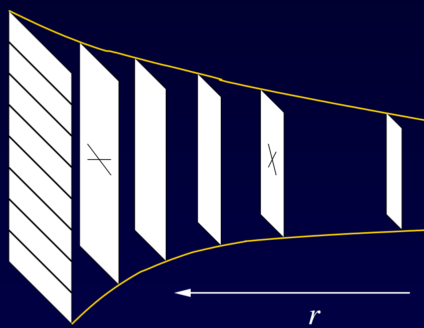
Lifshitz holography

AdS:



$$ds^2 = r^2(-dt^2 + d\mathbf{x}^2) + dr^2/r^2$$

Lifshitz:



$$ds^2 = -r^{2z}dt^2 + r^2 d\mathbf{x}^2 + dr^2/r^2$$

Focus of this talk:
The role of HL gravity for Lifshitz spacetime.

Which theory does this space solve?

Not Einstein equations in the vacuum . . .

Two options:

(a) Keep theory relativistic, modify by inventing suitable matter; Lifshitz spacetime may be a solution when matter is excited.

(b) Modify gravity; Lifshitz spacetime may be a vacuum solution.

We will see that in both cases, ideas of anisotropic gravity play a central role.

IIa.
Lifshitz holography in relativistic gravity

Relativistic holography for Lifshitz space

Various matter sources possible, one of the simplest/most popular is a massive vector: [Taylor]

$$S = \frac{1}{16\pi G_N}\int dt\, d^D\mathbf{x}\, dr\sqrt{-G}(\mathcal{R}-2\Lambda)+\frac{1}{8\pi G_N}\int dt\, d^D\mathbf{x}\sqrt{-g}\mathcal{K}$$

$$-\frac{1}{4}\int dt\, d^D\mathbf{x}\, dr\sqrt{-G}(F_{\mu\nu}F^{\mu\nu}+2m^2A_\mu A^\mu).$$

To get Lifshitz with given z as solution, one must take

$$\Lambda = \frac{1}{2}\left[z^2+(D-1)z+D^2\right], \quad m^2 = Dz, \quad \langle A_t\rangle = \frac{2(z-1)}{z}.$$

Also, various string/M-theory embeddings now exist.

[Balasubramanian,Narayan; Gauntlett et al; . . .]

Anisotropic conformal infinity

Even if we embed Lifshitz space into relativistic gravity with matter, we need anisotropic scaling to define properly the asymptotic structure near "infinity".

Conformal infinity [Penrose]: Rescale the bulk metric on $\mathcal{M}$

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} \equiv \Omega(t, \mathbf{x}, r) g_{\mu\nu},$$

so that $\tilde{g}_{\mu\nu}$ now extends "past infinity" of $\mathcal{M}$. Then take $\overline{\mathcal{M}}$.
For Lifshitz spacetimes with $z > 1$: Conformal infinity is a line!
(This contradicts holographic expectations).

Anisotropic conformal infinity [PH & C. Melby-Thompson, 2009]:
For asymptotically foliated spacetimes, use anisotropic Weyl transformations instead. Anisotropic conformal infinity of Lifshitz spacetime is of codimension one, with a preferred nonrelativistic foliation. (As expected from holography.)

Holographic renormalization

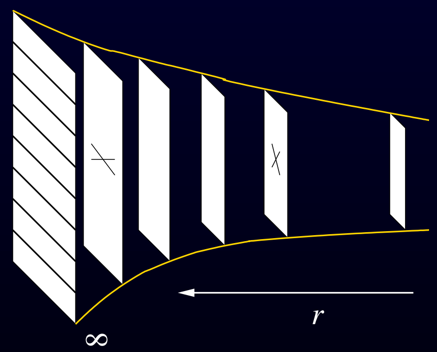
Holographic correspondence:

$$\mathcal{Z}_{\text{bulk}}\left[\Phi|_{\partial\mathcal{M}} = \Phi_0\right] = \left\langle \exp\left(\int_{\partial\mathcal{M}} \Phi_0 \mathcal{O}\right)\right\rangle_{\text{CFT}}.$$

In the classical gravity limit, this implies

$$W_{\text{eff}}[\Phi_0] = -S_{\text{on-shell}}[\Phi_0]_{\text{gravity}}.$$

The action diverges, requires (holographic) regularization and renormalization:



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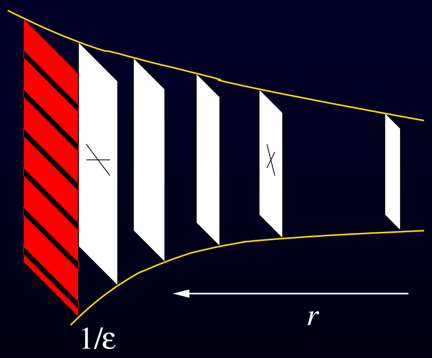

Counterterms: Hamilton-Jacobi theory

[de Boer, Verlinde, Verlinde; Skenderis et al; Ross & Saremi; . . .]

On-shell action $S[g_{ij}, N_i, N; \ldots]$, as a functional of boundary fields, satisfies HJ equation in radial evolution.

In gravity, this means $\mathcal{H}_r \left(\pi_{\alpha\beta} = \frac{\delta S}{\delta g_{\alpha\beta}} \right) = 0$ (plus supermomentum constraints).

Parametrize $S[g_{ij}, N_i, N] = \frac{1}{16\pi G_N} \int dt \, d^D\mathbf{x} \sqrt{g} N \, \mathcal{L}$.

Note: $\mathcal{L}$ is divergent, expand asymptotically as

$$\mathcal{L} = \ldots + \frac{\mathcal{L}^{(4)}}{\varepsilon^4} + \frac{\mathcal{L}^{(2)}}{\varepsilon^2} + \tilde{\mathcal{L}}^{(0)} \log \varepsilon + \mathcal{L}^{(0)} + \mathcal{O}(\varepsilon^2).$$

Plug into HJ equation, which gives a recursive relation among counterterms.

Counterterms for $z = 2$ in $D = 2$: Conformal Lifshitz gravity

Holographic recursion relations for counterterms can be solved for general D and z, at least in principle.

In the first nontrivial conformal case, $D = 2$ and $z = 2$, they give:

$$\mathcal{L}^{(4)} = 6, \quad \mathcal{L}^{(2)} = \frac{1}{2}R + \frac{1}{4}\left(\frac{\nabla N}{N}\right)^2, \quad \tilde{\mathcal{L}}^{(0)} = K_{ij}K^{ij} - \frac{1}{2}K^2.$$

The recursion relations imply $\delta_D \mathcal{L}^{(0)} \sim \tilde{\mathcal{L}}^{(0)}$.
Since $\mathcal{L}^{(0)}$ is the renormalized action, and δ_D (=the radial evolution operator) is the anisotropic Weyl rescaling,
$\tilde{\mathcal{L}}^{(0)}$ gives the anisotropic Weyl anomaly.
This anomaly takes the form of $z = 2$ conformal Lifshitz gravity action in detailed balance, at the anisotropic boundary.

Holographic anomaly in detailed balance

Note that of the two possible central charges, only c_K is generated in this holographic theory.
What distinguishes c_K from c_V? The c_K anomaly satisfies the detailed balance condition.

This was further verified for bulk gravity coupled to scalars X^I.
In addition to the kinetic piece,

$$\frac{1}{\kappa^2}\int dt\, d^2\mathbf{x}\sqrt{g}N\left(K_{ij}K^{ij}-\frac{1}{2}K^2\right)+\frac{1}{2}\int dt\, d^2\mathbf{x}\frac{\sqrt{g}}{N}\left(\dot{X}^I-N^i\partial_i X^I\right)^2,$$

the anomaly now develops also a potential part,

$$\int dt\, d^2\mathbf{x}\sqrt{g}N\left\{(\Delta X^I)^2+\frac{\kappa^2}{2}T_{ij}(X)T^{ij}(X)\right\}.$$

This is in detailed balance, with $W=\frac{1}{2}\int d^2\mathbf{x}\sqrt{g}\partial_i X^I\partial^i X^I$.

Field-theory examples with $z = 2$, $D = 2$

Lifshitz scalar field in gravitational background:

$$S = \frac{1}{2}\int dt\, d^2\mathbf{x}\sqrt{g}\left\{\frac{1}{N}\left(\dot{\Phi} - N^i\nabla_i\Phi\right)^2 - N(\Delta\Phi)^2\right\}.$$

Classically anisotropically Weyl invariant with $z = 2$, $\delta\Phi = 0$.

Quantum Weyl anomaly: [de Boer et al., also us]
Calculated using ζ-function regularization,

$$c_K = \frac{1}{32\pi}, \qquad c_V = 0.$$

Only one non-zero central charge; anomalies in detailed balance!

Detailed balance from holographic recursion

Why in the world should the holographic Weyl anomaly satisfy detailed balance?

Answer: This follows from the holographic recursion relation among the counterterms!

Indeed, the HJ equation – expanded order by order in the conformal dimension – implies that

$$\tilde{\mathcal{L}}^{(0)} \sim \left(\frac{\delta \mathcal{L}^{(2)}}{\delta g_{\alpha\beta}} \right)^2 .$$

This is precisely the condition of detailed balance, with the quadratic counterterm $\mathcal{L}^{(2)}$ playing the role of W!

More QFT examples with $z = 2$, $D = 2$

We saw that the minimal Lifshitz scalar has one central charge $c_K \neq 0$ but the other one $c_V = 0$.

Are there theories with both independent central charges?

(if not, then we don't need to look for their holographic description . . .)

More QFT examples with $z = 2$, $D = 2$

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Are there theories with both independent central charges?
(if not, then we don't need to look for their holographic description . . .)
Yes!

$$S = \frac{1}{2}\int dt\, d^2\mathbf{x}\sqrt{g}\left\{\frac{1}{N}\left(\dot{\Phi} - N^i\nabla_i\Phi\right)^2 - N(\Delta\Phi)^2\right\}$$
$$-\frac{e^2}{2}\int dt\, d^2\mathbf{x}\sqrt{g}N\left[R - \left(\frac{\nabla N}{N}\right)^2 + \frac{\Delta N}{N}\right]^2\Phi^2.$$

This non-minimally coupled theory has $c_K = 1/(32\pi)$ and $c_V = -e^2/(8\pi)$.

Now that we know that theories with independent c_K, c_V exist, what is the holographic dual of c_V?

Holography with the general anomaly?

- The simplest relativistic system does not work.

- If looking for a more complicated relativistic model:

Need to relax the holographic recursion between counterterms,

$$\tilde{\mathcal{L}}^{(0)} \neq \left(\frac{\delta \mathcal{L}^{(2)}}{\delta g_{\alpha\beta}}\right)^2 .$$

- But: it turns out that Lifshitz gravity works, in the vacuum!

Disclaimer: That does not imply that one cannot do this with relativistic gravity coupled to matter; indeed, the boundary between relativistic and nonrelativistic is somewhat fuzzy.)

IIb.
Lifshitz holography with Lifshitz gravity

Lifshitz gravity for Lifshitz holography

Requires nonprojectable theory (because we anticipate anisotropic Weyl transformations on the codimension-one boundary, hence N near boundary must depend on spacetime).

Lifshitz spacetime now foliated not just asymptotically, but everywhere (in the bulk). A single, codimension-one foliation by leaves of constant t (multiple & nested foliations also possible, not studied here).

At low energies, just as in GR:

• Anisotropic gravity will be dominated by $z = 1$ terms.

• We neglect the small corrections due to all other higher-derivative terms, including the $z > 1$ terms in S_V.

Low-energy effective theory

The low-energy action will be similar to Einstein-Hilbert, with several important differences:

$$S = \frac{1}{\kappa^2}\int dt\, d^D\mathbf{x}\, dr \sqrt{G} N \left(K_{ab}K^{ab} - \lambda K^2\right)$$

$$+\frac{1}{\kappa^2}\int dt\, d^D\mathbf{x}\, dr \sqrt{G} N \left[\beta(R-2\Lambda) - \frac{\alpha^2}{2}\left(\frac{\nabla_a N}{N}\right)^2\right].$$

We have three additional low-energy couplings compared to GR: λ, β and α.

Roughly, one will be fixed by choosing z for the vacuum, the other is free to give the missing central charge in the anomaly.

Lifshitz as the vacuum solution

Require now that the Lifshitz spacetime with chosen value of z is a solution of the vacuum EoM.

This determines the cosmological constant
(essentially, by our choice of scale),

$$\Lambda = -\frac{(D+z)(D+z-1)}{2},$$

and fixes one of the two new couplings,

$$\alpha^2 = \frac{2\beta(z-1)}{z},$$

leaving λ undetermined.

Note: This relation implies $\alpha^2 \leq 2\beta D/(D-1)$, a relation known in nonprojectable HL gravity.

Scalar gravitons & anisotropic BF bound

Although no extra matter field introduced, the scalar graviton plays the role of an extra propagating DoF.

One can ask for the scaling dimensions Δ_+ and Δ_- associated with the asymptotic behavior of the bulk scalar graviton:

$$\Delta_\pm = \frac{z+D}{2}\left\{1 \pm \sqrt{1+\frac{4(1-z)D}{(1-\lambda)(z+D)^2}}\right\}.$$

The requirement that $\Delta_\pm$ be real gives constraints on λ; for $z > 1$,

$$\lambda \geq 1 \quad \text{or} \quad \lambda \leq \lambda_U \equiv \frac{(z-D)^2+4D}{(z+D)^2}.$$

In the special case of $z = D$, the latter gives $\lambda \leq 1/D$, and opens up a BF-like window: $\dfrac{1}{D+1} \leq \lambda \leq \dfrac{1}{D}$.

Holographic recursion in HL gravity: The counterterms and the anomaly

Again, use the radial Hamilton-Jacobi formulation to perform holographic renormalization. Logic is identical to the relativistic case.

In the special case of $D = 2$, $z = 2$, we found the general form of the anisotropic Weyl anomaly:

$$\frac{1}{2\kappa^2}\left(K_{ij}K^{ij} - \frac{1}{2}K^2\right) + \frac{\beta}{48\kappa^2}\left[R - \left(\frac{\nabla N}{N}\right)^2 + \frac{\Delta N}{N}\right]^2 .$$

Thus, both c_K and c_V are independently produced in holographic renormalization of minimal HL gravity in the vacuum.

III.
Black holes & Conclusions

Black holes and holography at $T \neq 0$

Having shown that HL gravity represents the minimal model of holography for QFTs with Lifshitz scaling, one can try to thermalize the duality.

Saddle points at finite T: Black holes in HL gravity? Horizons? Temperature? Entropy? Two classes of solutions found:

(1) Simple Lifshitz-Schwarzschild-Painlevé black holes

$$N = r^z, \quad ds_\Sigma^2 = \frac{dr^2}{r^2} + r^2 d\mathbf{x}^2, \quad N_r = \frac{C}{r^{(D-z)/2+1}}.$$

These exist only when $\lambda = \lambda_U$, and have $M \propto C^2(D+2-z)$!

(2) Static solutions: Lifshitz-Schwarzschild fall-off, but analytically much more complicated ...

This is an interesting Gedanken laboratory, in both directions!

Conclusions

• Gravity with anisotropic scaling is finding multiple use in holography for nonrelativistic QFTs.

• Even when the bulk theory is relativistic, anisotropic Weyl symmmetry is crucial for the proper definition of conformal infinity of spacetime, and in the treatment of asymptotic expansions for holographic renormalization.

• Anisotropic Weyl symmetries can be anomalous. For $z = 2$ in $D = 2$, there are two independent central charges. Both can be independently realized in families of consistent QFTs.

• The minimal relativistic model of Lifshitz holography does not yield the most general form of anisotropic Weyl anomaly: Only one of the central charges is nonzero.

• Lifshitz spacetimes also solve vacuum EoM of nonprojectable HL gravity. This construction yields the most general form of the holographic anisotropic Weyl anomaly.

• HL gravity is the natural arena for nonrelativistic holography.