HW 1 SOLUTIONS

Problem 1

Writing out $\mathbf{v} \times \mathbf{w}$ in components, we have

$$\mathbf{v} \times \mathbf{w} = (v_y w_z - w_y v_z, v_z w_x - v_x w_z, v_y w_z - v_z w_y) \tag{1}$$

so taking the dot product with $\mathbf{z}$ gives

$$(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{z} = z_x(v_y w_z - w_y v_z) + z_y(v_z w_x - v_x w_z) + z_z(v_y w_z - v_z w_y) \tag{2}$$

which is exactly what we'd get by expanding out the determinant.

Problem 2

This problem is mostly just matrix multiplication, so we'll do little more than quote the answers.

a)We have

$$AB = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 9 \\ 5 & 3 & 3 \end{pmatrix} \tag{3}$$

so $|AB| = 1(-6-27) - 1(-6-3) + 5(-18+2) = -104$.

b)$AC = \begin{pmatrix} 9 & 7 \\ 13 & 9 \\ 5 & 2 \end{pmatrix}$

c)$ABC = \begin{pmatrix} -5 & -5 \\ 3 & -5 \\ 25 & 14 \end{pmatrix}$

d) $AB - B^t A^t = AB - (AB)^t = \begin{pmatrix} 0 & -3 & -4 \\ 3 & 0 & 6 \\ 4 & -6 & 0 \end{pmatrix}$

Problem 3

Our Equation of Motion is

$$m\ddot{y} + \alpha\dot{y} = -mg \tag{4}$$

where $\alpha > 0$. This equation is both separable and an inhomogenous constant coefficient linear ODE (ICCLODE for short), and thus has two standard methods of solution (remember that ODE course you took and then promptly forgot about?). We will view this equation as an ICCLODE and solve it accordingly.

First, we solve the homogenous equation

$$m\ddot{y} + \alpha\dot{y} = 0 \tag{5}$$

by making the usual ansatz $y_h = A\mathrm{e}^{rt}$ which, when plugged in to (5), yields the following *algebraic* equation for r:

$$mr^2 + \alpha r = 0, \tag{6}$$

with solutions $r = 0, -\frac{\alpha}{m}$, so the general solution to the homogenous equation is

$$y_h(t) = A\mathrm{e}^{-\frac{\alpha}{m}t} + B \tag{7}$$

where A and B are undetermined constants. Now we just need a particular solution to the inhomogenous equation (4). A little trial and error reveals that $y_p(t) = -\frac{mg}{\alpha}t$ is such a solution, so adding them together gives the general solution to the *inhomogenous* equation,

$$y(t) = A\mathrm{e}^{-\frac{\alpha}{m}t} + B - \frac{mg}{\alpha}t. \tag{8}$$

Setting $y(0) = h$ and $\dot{y}(0) = v$ yields two algebraic equations in A and B which can be solved to give, finally,

$$y(t) = h + \frac{m}{\alpha}(v + \frac{mg}{\alpha})(1 - \mathrm{e}^{-\frac{\alpha}{m}t}) - \frac{mg}{\alpha}t. \tag{9}$$

Problem 4

Without air resistance, the equations of motion are easily integrated to give

$$x(t) = v_x t \tag{10}$$

$$y(t) = -\frac{1}{2}gt^2 + v_y t. \tag{11}$$

To find the time at which the particle hits the ground we set $y(t_{ground}) = 0$ and find

$$t_{ground} = \frac{2v_y}{g} \tag{12}$$

which we then plug into our expression for $x(t)$ to find that range is $2\frac{v_x v_y}{g}$.

To include air resistance we note that our vector EOM

$$m\ddot{\mathbf{x}} + \alpha\dot{\mathbf{x}} + mg\hat{\mathbf{y}} = 0 \tag{13}$$

decouples into

$$m\ddot{x} + \alpha\dot{x} = 0 \tag{14}$$

$$m\ddot{y} + \alpha\dot{y} + mg = 0 \tag{15}$$

so borrowing our result from problem 3 and imposing slightly different initial conditions we have

$$y(t) = \frac{m}{\alpha}(v_y + \frac{mg}{\alpha})(1 - \mathrm{e}^{-\frac{\alpha}{m}t}) - \frac{mg}{\alpha}t, \tag{16}$$

a transcendental equation in t which we cannot solve analytically to find t_{ground}.

Problem 5

A one-liner: defining a distance function in such a way would depend highly on the *choice* of chart that you make, and is hence not a sensible definiton since functions must have *unique* values.