HW 10 SOLUTIONS

Problem 1

H&F Chapter 7, problem 2

To see that $\vec{\omega}$ transforms as a pseudovector (i.e. doesn't change sign under inversion), recall that $\vec{\omega} = (\omega_x, \omega_y, \omega_z)$ really comes from the antisymmetric tensor

$$A = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \tag{1}$$

so to investigate the transformation properties of $\vec{\omega}$ we must really investigate the transformation properties of A. But we know that under a transformation represented by an orthogonal transformation R, we have

$$A' = RA\tilde{R} \tag{2}$$

in terms of matrix multiplication. For inversion $R = -I$, so

$$A' = (-I)A(-I) = A \tag{3}$$

so A, and hence $\vec{\omega}$, are invariant under inversion.

Alternatively, one could note that for

$$(v_1, v_2, v_3) = (\omega_1, \omega_2, \omega_3) \times (r_1, r_2, r_3) \tag{4}$$

to hold in the inverted frame the w_i must not change sign, since the v_i and r_i will.

Problem 2

H&F Chapter 8, Problem 4

We are to compute the principal moments of inertia of a circle of mass M. Assume the circle lies in the x-y plane. Then clearly

$$I_{zz} = MR^2 \tag{5}$$

We also have

$$
\begin{aligned}
I_{xx} &= \oint y^2 dm & (6)\\
&= \oint y^2 \rho \, ds & (7)\\
&= 4\int_0^R (r^2 - x^2)\left(\frac{M}{2\pi R}\right)\sqrt{1 + \frac{x^2}{R^2 - x^2}}\, dx & (8)\\
&= \frac{2M}{\pi}\int_0^R \sqrt{R^2 - x^2}\, dx & (9)\\
&= \frac{MR^2}{2} & (10)
\end{aligned}
$$

which is of course the same as I_{yy} by symmetry.

H&F Chapter 8, Problem 5

Assume the rod lies along the x-axis with center of mass at the origin as usual. Then we have

$$
\begin{aligned}
I_{xx} &= 0 & (11)\\
I_{yy} &= 2\int_0^{\frac{l}{2}} \frac{M}{l} x^2 \, dx = \frac{Ml^2}{12} & (12)\\
I_{zz} &= I_{yy} = \frac{Ml^2}{12} & (13)
\end{aligned}
$$

Problem 3

H&F Chapter 8 Problem 8

By symmetry all moments will be the same so we take the easiest route and compute I_{zz} in spherical coordinates:

$$
\begin{aligned}
I_{zz}(\text{spherical shell}) &= \oint_{S^2} \rho(x^2 + y^2)\, dA & (14)\\
&= \int_0^\pi \int_0^{2\pi} \left(\frac{M}{4\pi R^2}\right)(R^2 \sin^2\theta)(R^2 \sin\theta \, d\phi d\theta) & (15)\\
&= \frac{MR^2}{2}\int_0^\pi \sin\theta(1 - \cos^2\theta) d\theta & (16)\\
&= \frac{2}{3} MR^2 & (17)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_{zz}(\text{sphere}) &= \frac{M}{\frac{4}{3}\pi R^3}\int_0^R\int_0^\pi\int_0^{2\pi} r^4\sin^3\theta\, d\phi d\theta dr && (18)\\
&= \frac{2}{5}MR^2 && (19)
\end{aligned}
$$

H& F Chapter 8 Problem 11

Following the book's advice we first compute the moments about an origin located halfway along the baseline between the two masses m_1. The vector pointing from the CM to this point is $\vec{a} = -d\hat{y}$, where $d = \frac{m_2 h}{M}$. The off-diagonal elements of the inertia tensor cancel by symmetry and we have

$$
\begin{aligned}
I_{xx} &= m_2h^2 && (20)\\
I_{yy} &= 2m_1(\frac{a^2}{4}) && (21)\\
I_{zz} &= 2m_1(\frac{a^2}{4}) + m_2h^2 && (22)
\end{aligned}
$$

so the inertia tensor for this origin is

$$
I_{\vec{a}} = \begin{pmatrix} m_2h^2 & 0 & 0 \\ 0 & \frac{m_1a^2}{2} & 0 \\ 0 & 0 & \frac{m_1a^2}{2} + m_2h^2 \end{pmatrix} \qquad (23)
$$

Then the displaced axis theorem gives us

$$
\begin{aligned}
I_{CM} &= I_{\vec{a}} - M\left[d^2 I - \begin{pmatrix} 0 & 0 & 0 \\ 0 & d^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] && (24)\\
&= \begin{pmatrix} \frac{2m_1m_2h^2}{M} & 0 & 0 \\ 0 & \frac{m_1a^2}{2} & 0 \\ 0 & 0 & \frac{m_1a^2}{2} + \frac{2m_1m_2h^2}{M} \end{pmatrix} && (25)
\end{aligned}
$$

Problem 4

H&F Chapter 8 Problem 13

a)We have

$$KE = KE_{rot} + KE_{CM} = \frac{1}{2}mv_0^2 + \frac{1}{2}I_3\omega_0^2 \tag{26}$$

$$\vec{L} = I_3\vec{\omega_0} \tag{27}$$

b) We have

$$KE = KE_{rot} + KE_{CM} = \frac{1}{2}mv_0^2 + \frac{1}{2}I\omega_0^2 \tag{28}$$

$$\vec{L} = I\vec{\omega_0} \tag{29}$$

Problem 5

H&F Chapter 8 Problem 17

From the Lagrangian

$$L = T = \frac{I}{2}(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 \tag{30}$$

we have the following Euler-Lagrange equations, where we have used the fact that ϕ and ψ are cyclic:

$$\dot{\phi}\sin\theta(I\dot{\phi}\cos\theta - I_3(\dot{\psi} + \dot{\phi}\cos\theta)) = I\ddot{\theta} \tag{31}$$

$$\frac{d}{dt}(I\dot{\phi}\sin^2\theta + I_3\cos\theta(\dot{\psi} + \dot{\phi}\cos\theta)) = 0 \tag{32}$$

$$\frac{d}{dt}(I_3(\dot{\psi} + \dot{\phi}\cos\theta)) = 0. \tag{33}$$

We may immediately integrate (33) and (32) and then plug the results into (31) to get

$$I\ddot{\theta} = \dot{\phi}\sin\theta(I\dot{\phi}\cos\theta - L_3) \tag{34}$$

$$I\dot{\phi}\sin^2\theta + L_3\cos\theta = l \tag{35}$$

$$I_3(\dot{\psi} + \dot{\phi}\cos\theta) = L_3 \tag{36}$$

where comparison between (36) and the text's eqn (8.68) reveals that L_3 really is the z-component of the angular momentum (in the *body* frame) and the suggestive naming of the constant of integration in (35) will be justified shortly. In principle we could now solve (35) and (36) for $\dot{\phi}$ and $\dot{\psi}$ in terms of θ and then plug in to (34) but this yields a highly nonlinear ODE for θ. Instead, we follow the book and make a convenient choice of coordinates: since $\vec{L}$ is constant, we may choose our space axes so that

$$\vec{L} = L\hat{z}' \tag{37}$$

Now, since L_3 is the component of $\vec{L}$ along the body z-axis and the body z-axis makes an angle θ with the space z-axis, we have

$$L_3 = L\cos\theta \tag{38}$$

and since L and L_3 are constant (the latter by (33)) we have that $\cos\theta$, and hence θ, are constant as well. Now we could substitute (38) into (35) and (36) and solve for $\dot{\phi}, \dot{\psi}$ as functions of time, but first we should determine the relationship bewteen L and l. To do this, note that in the body frame (denoting the moment of inertia tensor by $\mathcal{I}$),

$$\vec{L}|_{body} = \mathcal{I}|_{body}\vec{\omega}|_{body} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I_3 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} I\omega_1 \\ I\omega_2 \\ I_3\omega_3 \end{pmatrix} \tag{39}$$

so if

$$U = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & \sin\theta\sin\phi \\ \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & -\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi & -\sin\theta\cos\phi \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{pmatrix} \tag{40}$$

is our rotation matrix, then we better have

$$\vec{L}|_{space} = U\vec{L}|_{body} = \begin{pmatrix} 0 \\ 0 \\ L \end{pmatrix}. \tag{41}$$

Using (40), (39) and the text's eqn (8.68) which gives the components of $\vec{\omega}|_{body}$ in terms of the Euler angle and their time derivatives, we can write out the third component of (41), which after some cancellations is

$$L = I\dot{\phi}\sin^2\theta + L_3\cos\theta = l. \tag{42}$$

Now we can solve (35) and (36) for $\dot{\phi}, \dot{\psi}$ in terms of L and $\cos\theta$, yielding

$$\begin{aligned}
\dot{\phi} &= \frac{L}{I} && (43) \\
\dot{\psi} &= L\cos\theta\left(\frac{1}{I_3} - \frac{1}{I}\right) && (44)
\end{aligned}$$

so then we finally have

$$\theta = \text{constant} \tag{45}$$

$$\phi = \frac{L}{I}t + \phi_0 \tag{46}$$

$$\psi = L\cos\theta(\frac{1}{I_3} - \frac{1}{I})t + \psi_0 \tag{47}$$

A little reflection on the definition of the Euler angles will show that $\dot{\psi}$ is Ω, the rate at which the top spins around the body z-axis, and that $\dot{\phi}$ is ω_p, the rate of precession, so we have

$$\begin{aligned}
\Omega &= L\cos\theta\left(\frac{1}{I_3} - \frac{1}{I}\right) && (48) \\
\omega_p &= \frac{L}{I} && (49)
\end{aligned}$$

which agrees with the text's eqns (8.46) and (8.47).