HW 11 SOLUTIONS

Problem 1

H&F Chapter 9, Problem 2

For our three pendula in a constant gravitational field, arranged linearly and each coupled to each other by springs of spring constant ϵ, we have the exact Lagrangian (setting $m = g = l = 1$ and setting the equilibirum lengths of the springs to 0 as discussed in section)

$$L = \frac{1}{2}\sum_{i=1}^{3}\dot{\theta_i}^2 - \sum_{i=1}^{3}(1-\cos\theta_i) - \frac{\epsilon}{2}[(\theta_1-\theta_2)^2 + (\theta_2-\theta_3)^2 + (\theta_1-\theta_3)^2] \quad (1)$$

which in the small-angle approximation reduces to

$$L = \frac{1}{2}\sum_{i=1}^{3}\dot{\theta_i}^2 - \frac{1}{2}\sum_{i=1}^{3}(1+2\epsilon)\dot{\theta_i}^2 + \epsilon(\theta_1\theta_2 + \theta_2\theta_3 + \theta_1\theta_3) \quad (2)$$

so we get

$$\mathbf{t} = \frac{1}{2}I \quad (3)$$

$$\mathbf{v} = \begin{pmatrix} 1+2\epsilon & -\epsilon & -\epsilon \\ -\epsilon & 1+2\epsilon & -\epsilon \\ -\epsilon & -\epsilon & 1+2\epsilon \end{pmatrix} \quad (4)$$

so we have (dropping constant factors in the determinant)

$$\begin{aligned} |\mathbf{v} - \omega^2\mathbf{t}| &= \begin{vmatrix} 1+2\epsilon-\omega^2 & -\epsilon & -\epsilon \\ -\epsilon & 1+2\epsilon-\omega^2 & -\epsilon \\ -\epsilon & -\epsilon & 1+2\epsilon-\omega^2 \end{vmatrix} & (5) \\ &= -\omega^6 + 3(1+2\epsilon)\omega^4 - 3(1+3\epsilon)(1+\epsilon)\omega^2 + (1+3\epsilon)^2 & (6) \\ &= (\omega^2-1)(-\omega^4 + (2+6\epsilon)\omega^2 - (1+3\epsilon)^2). & (7) \end{aligned}$$

We knew to factor out $\omega^2 - 1$ since we knew that the mode in which all the pendula swing in phase and with same amplitude has $\omega = 1$. In this mode the springs do not stretch at all (hence ϵ is not involved). Anyhow, Setting (7) equal to 0 yields

$$\omega^2 = 1,\ 1+3\epsilon \quad (8)$$

where the 2nd frequency is doubly degenerate. The mode associated to $\omega^2 = 1$ clearly has mode vector $(1,1,1)$. The other modes vectors are orthogonal to this one with respect to $\mathbf{t}$, but since $\mathbf{t}$ is proportional to the identity matrix this just means orthogonality in the usual sense, and since the other mode frequency is doubly degenerate we can choose any basis we want in the orthogonal complement to $(1,1,1)$, so we take the following set of mode vectors:

$$\omega^2 = 1 \qquad (1,1,1) \tag{9}$$
$$\omega^2 = 1+3\epsilon \quad (1,0,-1),\ (1,-2,1) \tag{10}$$

As for restrictions on ϵ, we see that in order for ω^2 to be real we must have

$$\epsilon > -\frac{1}{3} \tag{11}$$

but for springs the spring constant is always positive so this is automatically satisfied.

Problem 2

H&F Chapter 9 Problem 3

Our Lagrangian for the two masses connected to three springs is (again setting m=1)

$$\begin{aligned} L &= \frac{1}{2}(\dot{x_1}^2 + \dot{x_2}^2) + \frac{k}{2}(x_2 - x_1)^2 + \frac{1}{2}(x_1^2 + x_2^2) && (12) \\ &= \frac{1}{2}(\dot{x_1}^2 + \dot{x_2}^2) + \frac{1}{2}[(1+k)x_1^2 + (1+k)x_2^2 - 2kx_1x_2] && (13) \end{aligned}$$

so we have, again dropping constant factors,

$$\begin{aligned} |\mathbf{v} - \omega^2\mathbf{t}| &= \begin{vmatrix} k+1-\omega^2 & -k \\ -k & k+1-\omega^2 \end{vmatrix} && (14) \\ &= \omega^4 - 2(k+1)\omega^2 + 2k + 1 && (15) \end{aligned}$$

which has solutions

$$\omega^2 = 1,\ 2k+1. \tag{16}$$

To get the mode vectors we use Kramer's rule, which in this case yields $(k+1-\omega^2, k)$ so we have

$$\omega^2 = 1 \;\; (1,1) \tag{17}$$

$$\omega^2 = 2k+1 \;\; (1,-1) \tag{18}$$

Problem 3

H&F Chapter 9 Problem 4

In this problem we have 3 degrees of freedom, which we can take to be z, the height of the center of mass of the plane measured from the equilibrium point, and θ_x, θ_y, the angles of rotation about the x and y-axes respectively. Another way to see that there are three degrees of freedom is to note that the length of the 4 springs certainly determines the configuration of the system, but this is redundant since 3 points determine a plane so once one knows the length of three of the springs the length of the 4th is determined. Now, if we just have CM motion it's clear that the restoring force is $-4kz\hat{z}$, so the CM mode has mode frequency

$$\omega_{cm} = \sqrt{4k/M} \tag{19}$$

We neglected gravity since gravity serves only to change the equilibrium position of the plate. Now assume the CM is fixed, so then our Lagrangian is, assuming that the plane has dimensions a and b along the x and y axes respectively,

$$L = \frac{1}{2}[I_x\dot{\theta_x}^2 + I_y\dot{\theta_y}^2 - \tag{20}$$

$$2k\left(\left(\frac{a}{2}\sin\theta_y + \frac{b}{2}\sin\theta_x\right)^2 + \left(\frac{a}{2}\sin\theta_y - \frac{b}{2}\sin\theta_x\right)^2\right)] \tag{21}$$

$$\approx \frac{1}{2}\left[\frac{Mb^2}{12}\dot{\theta_x}^2 + \frac{Ma^2}{12}\dot{\theta_y}^2 - k(a^2\theta_y^2 + b^2\theta_x^2)\right] \tag{22}$$

where we used the fact that the heights of the various springs in terms of θ_x, θ_y are $\pm(\frac{a}{2}\sin\theta_y \pm \frac{b}{2}\sin\theta_x)$. From (22) we see that θ_x and θ_y decouple, and we can read off the mode frequencies from the Lagrangian as

$$\omega_{rot}^2 = \sqrt{\frac{12k}{M}} \tag{23}$$

(If you don't buy it just write down the EOM's from the Lagrangian). Note that the dimensions of the plate cancel out; although the moments of inertia grow larger as the dimensions do, the displacement of and torque exerted by the springs also grow, and these effects cancel each other out.

Problem 4

H&F Chapter 9 Problem 12

In terms of our original coordinates x_1, x_2, x_3 we have (adjusting x_1 and x_3 so that l drops out, as in the text)

$$L = \frac{m}{2}(\dot{x_1}^2 + \dot{x_2}^2) + \frac{M}{2}\dot{x_2}^2 + \frac{k}{2}[(x_1 - x_2)^2 + (x_2 - x_3)^2] \tag{24}$$

Now we change to 'relative' coordinates

$$x_1 = y_1 + x_{cm} \tag{25}$$
$$x_3 = y_3 + x_{cm} \tag{26}$$
$$x_2 = x_{cm} - \beta(y_1 + y_3) \tag{27}$$

where $\beta \equiv \frac{m}{M}$ and (27) is derived from (25),(26) and the definition of x_{cm}. Substituting these new coordinates into (24) yields, eventually,

$$L = \frac{1}{2}(2m + M)\dot{x_{cm}}^2 + \frac{m}{2}(1 + \beta)(\dot{y_1}^2 + \dot{y_3}^2) + m\beta\dot{y_1}\dot{y_3} + \tag{28}$$
$$\frac{k}{2}[(1 + 2\beta + 2\beta^2)(y_1^2 + y_3^2) + 4\beta(1 + \beta)y_1 y_3]. \tag{29}$$

Now we could write down $\mathbf{t}$ and $\mathbf{v}$ for this Lagrangian and diagonalize, but this ends up being quite messy, so instead we just note the symmetry in y_1 and y_3 and make the following substitution

$$p = y_1 + y_3 \tag{30}$$
$$q = y_1 - y_3 \tag{31}$$

in terms of which

$$y_1^2 + y_3^2 = \frac{1}{2}(p^2 + q^2) \tag{32}$$
$$y_1 y_3 = \frac{1}{4}(p^2 - q^2) \tag{33}$$

so our Lagrangian then becomes (ignoring the x_{cm} degree of freedom)

$$\begin{aligned} L &= \frac{m(1+\beta)}{4}(\dot{p}^2+\dot{q}^2)+\frac{m\beta}{4}(\dot{p}^2-\dot{q}^2)+ && (34)\\ &\quad \frac{k}{2}[\frac{1}{2}(1+2\beta+2\beta^2)(p^2+q^2)+\beta(1+\beta)(p^2-q_2)] && (35)\\ &= \frac{m}{4}(1+2\beta)\dot{p}^2 \ + \ \frac{m}{4}\dot{q}^2 \ + \ \frac{k}{4}[(1 \ + \ 2\beta)^2p^2 \ + \ q^2] && \\ & && (37) \end{aligned}$$

so we see that p and q decouple and we can just read off the frequencies. Along with the mode vectors expressed in terms of y_1, y_3 we have

$$\omega_p^2 = \sqrt{\frac{k}{m}(1+2\beta)} \quad (1,1) \tag{38}$$

$$\omega_q^2 = \sqrt{\frac{k}{m}} \quad (1,-1)$$

Problem 5

H&F Chapter 9 Problem 20

From the text's eqn (9.125) we have

$$\omega_1 = 2\sqrt{\frac{\tau}{md}}\sin\frac{\pi}{8} \tag{40}$$

$$\omega_2 = \sqrt{2\frac{\tau}{md}} \tag{41}$$

$$\omega_3 = 2\sqrt{\frac{\tau}{md}}\sin\frac{3\pi}{8} \tag{42}$$

and the corresponding mode vectors are

$$(\frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}), \quad (1,0,-1), \quad (\frac{1}{\sqrt{2}}, -1, \frac{1}{\sqrt{2}}) \tag{43}$$