HW 3 SOLUTIONS

Problem 1

First we show that if we have no time-dependent constraints (i.e. $\mathbf{r}_i = \mathbf{r}_i(q_1, \ldots, q_k)$ where i labels the different particles in the system) then T is a quadratic function of the $\dot{q}_k$'s. Note that we make no claim about the dependence of T on the q_k. As always, summation over repeated indices is implied. First we have

$$T = \frac{1}{2} m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i = \frac{1}{2} m_i \frac{\partial \mathbf{r}_i}{\partial q_k} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_k \dot{q}_j \tag{1}$$

by the chain rule. Then noticing that, since $\mathbf{r}_i = \mathbf{r}_i(q_1, \ldots, q_k)$ we also have $\frac{\partial \mathbf{r}_i}{\partial q_j} = \frac{\partial \mathbf{r}_i}{\partial q_j}(q_1, \ldots, q_k)$, we see that T is a quadratic (degree two) function of the $\dot{q}_k$'s.

That being the case, we have

$$p_l \dot{q}_l = \frac{\partial L}{\partial \dot{q}_l} \dot{q}_l = \frac{1}{2} m_i \dot{q}_l \frac{\partial \mathbf{r}_i}{\partial q_k} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} (\dot{q}_k \delta_{jl} + \dot{q}_j \delta_{kl}) = m_i \frac{\partial \mathbf{r}_i}{\partial q_k} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_k \dot{q}_j = 2T \tag{2}$$

so

$$H = p_k q_k - L = 2T - (T - V) = T + V = E \tag{3}$$

Problem 2

a)Since $y = f(x)$ we have by the chain rule $\dot{y} = \dot{x} f'$ so $T = \frac{1}{2} m (\dot{x^2} + \dot{y^2}) = \frac{1}{2} m \dot{x^2} (1 + f'^2)$ and thus

$$L = \frac{1}{2} m \dot{x^2} (1 + f'^2) - mgf \tag{4}$$

so

$$\begin{aligned} \frac{\partial L}{\partial x} &= m \dot{x^2} f'' - mgf' && (5) \\ \frac{\partial L}{\partial \dot{x}} &= m \dot{x} (1 + f'^2) && (6) \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} &= m \ddot{x} (1 + f'^2) + 2m \dot{x}^2 f'' f' && (7) \end{aligned}$$

and thus our EOM is

$$m\ddot{x}(1+f'^2)+m\dot{x}^2f''f'+mgf'=0$$

b)Assuming that the particle has no KE at the top of the wire, we have

$$H=mgf(0)=mgf(x)+\frac{1}{2}m\dot{x}^2(1+f'^2) \tag{9}$$

so solving for $\dot{x}$ yields

$$\dot{x}=\sqrt{\frac{2g(f(0)-f(x))}{1+f'^2}}$$

c)

$$\tau=\int_0^\tau dt=\int_0^1\frac{dt}{dx}dx=\int_0^1\frac{1}{\dot{x}}dx \tag{11}$$

so $h(x)=\sqrt{\frac{1+f'^2}{2g(f(0)-f(x))}}$.

Problem 3

We have

$$L=\frac{m}{2}(\dot{r}^2+r^2\dot{\theta}^2)-V(r) \tag{12}$$

so

$$p_\theta=mr^2\dot{\theta}\equiv J \tag{13}$$

where J is a constant since $\frac{\partial L}{\partial\theta}=0$. Thus

$$\dot{\theta}=\frac{J}{mr^2} \tag{14}$$

and we can try plugging this into our old Lagrangian to get a new Lagrangian

$$L'=\frac{m}{2}\dot{r}^2+\frac{J^2}{2mr^2}-V(r) \tag{15}$$

One easily finds the E-L equation for r to be

$$m\ddot{r}=-\frac{dV}{dr}-\frac{J^2}{mr^3} \tag{16}$$

which does *not* agree with eqn. 1.77 in the text. Instead, if we consider the Routhian

$$R = R(r, \dot{r}, \theta, \dot{\theta} = \frac{J}{mr^2}) = L - p_\theta \dot{q}_\theta \tag{17}$$

$$= \frac{m}{2}\dot{r}^2 + \frac{J^2}{2mr^2} - V(r) - J\frac{J}{mr^2} \tag{18}$$

$$= \frac{m}{2}\dot{r}^2 - \frac{J^2}{2mr^2} - V(r) \tag{19}$$

then our E-L eqn is

$$\ddot{r} = -\frac{dV}{dr} + \frac{J^2}{mr^3} \tag{20}$$

as desired.

Problem 4

Given a curve $\gamma(t) = (r(t)cos\theta(t), r(t)sin\theta(t))$, $t \in [0, 1]$, one easily computes that

$$\dot{\gamma}^2 = \dot{r}^2 + r^2\dot{\theta}^2. \tag{21}$$

We then consider the integral

$$A = \int_0^1 \dot{\gamma}^2 dt = \int_0^1 (\dot{r}^2 + r^2\dot{\theta}^2) dt \tag{22}$$

This is of course not quite the same as the length integral L (which would have the integrand in a square root) but one can show that the ODE obtained from A yields the same paths one would get from L but with the additional constraint that $\dot{\gamma}$ be constant. So we apply our variational principles to A and get the following two coupled ODE, which just say (in polar coordinates) that γ should be a zero acceleration path:

$$\ddot{r} - r\dot{\theta}^2 = 0 \tag{23}$$

$$\frac{d}{dt}(r^2\dot{\theta}) = 0 \tag{24}$$

The second equation above can be integrated trivially to get

$$r^2\dot{\theta} = k \tag{25}$$

where k is a constant. Noting that the curve we seek passes through the origin ($r = 0$), we see from (25) that, in our case (but certainly not in others), $k = 0$. Then, again from (25), we see that for $r \neq 0$ we have $\dot{\theta} = 0$ so (23) tells us that $\ddot{r} = 0$ and hence

$$\gamma(t) = (atcos(\theta_0), atsin(\theta_0)). \tag{26}$$

where a, θ_0 are constants. Demanding that $\gamma(1) = (1, 1)$ fixes a and θ_0, yielding

$$\gamma(t) = (t, t) \tag{27}$$

Problem 5

a) Since we are taking the positive x-direction to be downwards and we are taking x=0 at the top of the path, conservation of energy reads

$$\frac{1}{2}mv^2 = mgx \tag{28}$$

or

$$v = \sqrt{2gx} \tag{29}$$

and this combined with $\frac{ds}{dx} = \sqrt{1 + y'^2}$ yields for the time T

$$T[y(x)] = \int_{x_0}^{x_1} \frac{1}{\sqrt{2gx}} \sqrt{1 + y'^2} \tag{30}$$

which gives the eqn 2.76 from the text when multiplied on both sides by $\sqrt{2g}$.

b)The integrand in T is

$$L = L(y, y', x) = \sqrt{\frac{1 + y'^2}{x}} \tag{31}$$

so since $\frac{\partial L}{\partial y} = 0$ we have

$$\frac{\partial L}{\partial y'} = \frac{y'}{\sqrt{x(1 + y'^2)}} = \sqrt{r} \tag{32}$$

where r is a constant. This can be solved to give

$$y' = \sqrt{\frac{xr}{1 - xr}} \tag{33}$$

which can be integrated by parts to yield, with the condition x(0)=0,

$$y(x) = -\frac{1}{2\sqrt{r}}\sqrt{x(1-rx)} + \frac{1}{2r}sin^{-1}(\sqrt{rx}) \tag{34}$$

which differs slightly from eqn. 2.77 in the text.