HW 4 SOLUTIONS

Problem 1

a) The position vector of our particle in terms of t and our generalized coordinate θ is, taking the center of the ring to be the origin and identifying Ωt as our azimuthal angle and θ as a cousin of the usual polar angle in spherical coordinates,

$$\mathbf{r}(\theta, t) = (Rcos(\Omega t)sin(\theta), Rsin(\Omega t)sin(\theta), -Rcos(\theta)) \tag{1}$$

so taking a time derivative and a dot product yields

$$T = \frac{m}{2}R^2(\dot{\theta}^2 + \Omega^2 sin^2\theta) \tag{2}$$

Combining this with the usual $V = mgR(1 - cos\theta)$ yields

$$L = \frac{m}{2}R^2(\dot{\theta}^2 + \Omega^2 sin^2\theta) - mgR(1 - cos\theta) \tag{3}$$

b)We have

$$\frac{\partial L}{\partial \theta} = -mgRsin\theta + mR^2\Omega^2 sin\theta cos\theta \tag{4}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = mR^2\ddot{\theta} \tag{5}$$

so by the E-L equations the equilbrium condition $\ddot{\theta} = 0$ is equivalent to

$$R\Omega^2 cos\theta sin\theta = gsin\theta \tag{6}$$

Note that we do not divide both sides by $sin\theta$ since $sin\theta$ might be 0. In fact, $\theta = 0, \pi$ satisfy (6) and hence give us two equilibrium points. If $\theta \neq 0, \pi$ then we may divide (6) by $sin\theta$ to get the following equation for an additional equilibrium point θ_0:

$$cos\theta_0 = \frac{g}{R\Omega^2} \tag{7}$$

which has a solution (other than 0 or π) if and only if

$$\frac{g}{R\Omega^2} < 1 \Leftrightarrow \Omega > \sqrt{g/R} \equiv \Omega_{crit} \tag{8}$$

c)We consider our equilibirum points one at a time.

i)θ =0

For $\theta - 0 = \theta << 1$, we have up to order θ^2

$$sin\theta \sim \theta \tag{9}$$
$$cos\theta \sim 1 - \theta^2/2 \tag{10}$$

so plugging these into L gives

$$L = \frac{m}{2}(R^2\dot{\theta}^2 + R^2\Omega^2\theta^2) - mgR\theta^2/2 = mR/2[R\dot{\theta}^2 + (R\Omega^2 - g)\theta^2] \tag{11}$$

so this equilibrium point is unstable if $R\Omega^2 > g$. Physically, this just means that if the hoop is spinning fast enough and one slightly displaces the bead from the bottom, centrifugal forces will overcome gravity and pull the bead outward, and hence upward.

ii)$\theta = \pi$

Now we have to expand around π. We have

$$\begin{aligned} cos\theta &\sim cos\theta|_\pi + \frac{d}{d\theta}cos\theta|_\pi(\theta-\pi) + \frac{d^2}{d\theta^2}cos\theta|_\pi(\theta-\pi)^2 && (12)\\ &= -1 + \frac{1}{2}(\theta-\pi)^2 && (13)\\ sin\theta &\sim sin\theta|_\pi + \frac{d}{d\theta}sin\theta|_\pi(\theta-\pi) + \frac{d^2}{d\theta^2}sin\theta|_\pi(\theta-\pi)^2 && (14)\\ &= \pi - \theta && (15) \end{aligned}$$

To clean things up a bit introduce $\phi \equiv \theta - \pi$, the displacement from equilibrium. This corresponds to setting our generalized coordinate to 0 at equilibrium, as discussed in the text. In terms of ϕ, we have

$$\begin{aligned} cos\theta &\sim -1 + \frac{1}{2}\phi^2 && (16)\\ sin\theta &\sim -\phi && (17)\\ sin^2\theta &\sim \phi^2 && (18) \end{aligned}$$

So our Lagrangian is, noting that $\dot{\phi} = -\dot{\theta}$,

$$L = \frac{m}{2}(R^2\dot{\phi}^2 + R^2\Omega^2\phi^2) - 2mgR + \frac{mgR}{2}\phi^2 = \frac{m}{2}R^2\dot{\phi}^2 + \frac{mR}{2}(R\Omega^2 + g)\phi^2 \tag{19}$$

where we have thrown out any constants in the Lagrangian since we know that they don't influence the eqns of motion. We see that $\theta = \pi$ is always unstable since the coefficient of ϕ^2 in (19) is always positive. This makes sense since both the centrifugal force and gravity act to pull the bead down from the top.

iii)$\theta_0 = cos^{-1}(\frac{g}{R\Omega^2})$
Here things get a little tricky. We follow the exact same logic as above, but the computation gets a little hairier, and we won't present all the details. To clean things up, introduce $x \equiv \frac{g}{R\Omega^2} = (\frac{\Omega_{crit}}{\Omega})^2$ and $\psi \equiv \theta - \theta_0$. Then we have, using $sin(cos^{-1}(x)) = \sqrt{1-x^2}$

$$cos\theta \sim x - \sqrt{1-x^2}\psi - \frac{x}{2}\psi^2 \tag{20}$$

$$sin\theta \sim \sqrt{1-x^2} + x\psi - \frac{1}{2}\sqrt{1-x^2}\psi^2 \tag{21}$$

$$sin^2\theta \sim 1 - x^2 + 2x\sqrt{1-x^2}\psi + (2x^2-1)\psi^2 \tag{22}$$

where (22) was obtained by very carefully squaring (21) and keeping all terms up to second order. Plugging all this into the Lagrangian, we find that the terms linear in ψ cancel (why should we expect this?) and we are eventually left with

$$L = const + \frac{m}{2}R^2\dot{\psi}^2 + \frac{mR}{2}(2R\Omega^2x^2 - R\Omega^2 - gx)\psi^2 \tag{23}$$

Plugging in our definition of x, we find the coefficient of ψ^2 in (23) to be $-\frac{mR^2\Omega^2}{2}(1 - \frac{g^2}{R^2\Omega^4})$ which is always negative (the expression in parentheses never becomes negative itself because this equilibirum point only exists when $x < 1$ which is equivalent to the expression in parentheses being positive). Thus this equilibrium point, when it exists, is always stable.

Problem 2

For the driven HO, we have

$$L = T - V + F(t)q \tag{24}$$

and since we have no constraints at all, let alone time-dependent ones, the result of Problem 1 of HW 3 applies and we have

$$H = 2T - L = T + V - F(t)q \neq T + V = E \tag{25}$$

For the pendulum of changing length we have, using $\mathbf{r}(t) = (l(t)sin\theta, -l(t)cos\theta))$ (where the origin is at the pivot of the pendulum),

$$L = \frac{m}{2}(l^2(t)\dot{\theta}^2 + \dot{l}(t)^2) - mgl(t)(1 - cos\theta) \tag{26}$$

so

$$H = \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L = \frac{m}{2}(l^2(t)\dot{\theta}^2 - \dot{l}(t)^2) + mgl(t)(1 - cos\theta) \tag{27}$$

which can't be the energy since the $\dot{l}(t)^2$ term has the wrong sign.

Problem 3

If ω is the frequency when m_1 is held fixed then, using the fact that $\omega = \sqrt{k/m_2} \Leftrightarrow k = \omega^2 m_2$ we have

$$w_{m_2 fixed} = \sqrt{k/m_1} = \omega\sqrt{m_2/m_1}. \tag{28}$$

One can derive this more formally by writing down the Lagrangian in both cases and including only the kinetic term for the mass which isn't fixed.

Now, if both masses are free we then have 2 degrees of freedom, one of which will correspond to the Center of Mass which we will set to 0 and the other of which will correspond to the relative distance between the masses. More precisely, we write L in terms of the coordinates x_1 and x_2 of the individual masses defined with respect to some origin

$$L = 1/2(m_1\dot{x_1}^2 + m_2\dot{x_2}^2) + k/2(x_1 - x_2)^2 \tag{29}$$

and then introduce new coordinates

$$Q = \frac{1}{m_1 + m_2}(m_1 x_1 + m_2 x_2) \tag{30}$$

$$q = x_1 - x_2 \tag{31}$$

where clearly Q is the CM coordinate and q is the relative coordinate. Now we may assume that $Q = \dot{Q} = 0$; if not, we just move to the frame in which this is true (this will be another inertial frame since the CM moves uniformly). Now, one can check that

$$\dot{Q} = 0 \Rightarrow \dot{x_2} = -\frac{m_1}{m_2}\dot{x_1} \tag{32}$$

which then implies that

$$\dot{q} = \dot{x_1} - \dot{x_2} = (1 + \frac{m_1}{m_2})\dot{x_1}. \tag{33}$$

Using (33) and (32) we can then rewrite L solely in terms of q and $\dot{q}$, yielding (after a little algebra)

$$L = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\dot{q}^2 + \frac{k}{2}q^2 = \frac{\mu}{2}\dot{q}^2 + \frac{k}{2}q^2 \tag{34}$$

where $\mu \equiv \frac{m_1 m_2}{m_1+m_2}$ is known as the *reduced mass*. Then we finally have

$$\omega_{free} = \sqrt{k/\mu} = \sqrt{\omega^2 m_2(\frac{1}{m_1} + \frac{1}{m_2})} = \omega\sqrt{1 + \frac{m_2}{m_1}}. \tag{35}$$

Problem 4

Using $Q = \frac{1}{2} - \epsilon$, eqn (3.35) from the text, and the first order taylor expansion $(1+x)^n \sim 1 + nx$ we have

$$\begin{aligned}
\lambda_\pm &= -\frac{1}{1-2\epsilon} \pm \sqrt{\frac{1}{4(\frac{1}{2}-\epsilon)^2} - 1} && (36)\\
&\sim -(1+2\epsilon) \pm \sqrt{\frac{1}{(1-2\epsilon)^2} - 1} && (37)\\
&\sim -1 - 2\epsilon \pm \sqrt{(1+4\epsilon) - 1} && (38)\\
&= -1 - 2\epsilon \pm 2\sqrt{\epsilon} && (39)
\end{aligned}$$

Thus

$$\begin{aligned}
e^{\lambda_+ t} - e^{\lambda_- t} &= e^{(-1-2\epsilon)t}(e^{2\sqrt{\epsilon}t} - e^{-2\sqrt{\epsilon}t}) && (40)\\
&= 2e^{(-1-2\epsilon)t} sinh(2\sqrt{\epsilon}t) && (41)\\
&\overset{\epsilon<<1}{\sim} 2e^{-t}(2\sqrt{\epsilon}t) && (42)\\
&= 4t\sqrt{\epsilon}e^{-t} && (43)\\
&\Rightarrow D = 4\sqrt{\epsilon} && (44)
\end{aligned}$$

Problem 5

a) For $t > 0$, we have

$$\ddot{q} + 2\dot{q} + q = 1 \tag{45}$$

To find the genereal solution, we first note that $q = 1$ is a particular solution. Then adding to that the known general solution to the homogeneous version of (45) yields

$$q(t) = Ae^{-t} + Bte^{-t} + 1 \tag{46}$$

Applying the boundary conditions $q(0) = \dot{q}(0) = 0$ yields $A = B = -1$, so

$$q(t) = -e^{-t} - te^{-t} + 1 \tag{47}$$

b)First we write down a complex EOM for a complex coordinate q_c, the real part of which is the EOM we really want to solve:

$$\ddot{q}_c + 2\dot{q}_c + q_c = e^{it}. \tag{48}$$

Plugging in the ansatz $q_c = Ae^{it}$ and dividing out by e^{it} yields

$$-A + 2iA + A = 1 \Rightarrow A = -i/2 \tag{49}$$

so we have a particular solution

$$q_p(t) = Re(q_c(t)) = Re(-\frac{i}{2}e^{it}) = \frac{1}{2}sint(50)$$

giving a relative phase of $\pi/4$.

c)Again, the general solution to the inhomogeneous equation is the sum of a particular solution (like the one found above) and the general solution to the homogeneous equation:

$$q(t) = Ae^{-t} + Bte^{-t} + \frac{1}{2}sint. \tag{51}$$

Applying our boundary conditions $q(0) = \dot{q}(0) = 0$ yields $A = 0, B = -1/2$ so

$$q(t) = 1/2(sint - te^{-t}). \tag{52}$$