HW 7 SOLUTIONS

Problem 1

HF Chapter 5, problem 4

In terms of z, we have (using $\theta = \frac{z}{k}$)

$$\mathbf{r}(t) = (rcos\frac{z}{k}, rsin\frac{z}{k}, z) \tag{1}$$

so the Lagrangian is

$$L = \frac{m}{2}(1 + \frac{r^2}{k^2})\dot{z}^2 - mgz \tag{2}$$

so the conjugate momentum is

$$p = m(1 + \frac{r^2}{k^2})\dot{z} \tag{3}$$

so we have

$$\dot{z} = \frac{p}{m(1 + \frac{r^2}{k^2})} \tag{4}$$

so the Hamiltonian is

$$H = \frac{p^2}{m(1 + \frac{r^2}{k^2})} - L = \frac{p^2}{2m(1 + \frac{r^2}{k^2})} + mgz \tag{5}$$

and Hamilton's eqns are therefore (4) and

$$\dot{p} = -\frac{\partial H}{\partial z} = -mg. \tag{6}$$

Substituting (4) into (6) yields

$$m(1 + \frac{r^2}{k^2})\ddot{z} = -mg \tag{7}$$

or

$$\ddot{z} = -\frac{g}{1 + \frac{r^2}{k^2}} \tag{8}$$

which can be immediately integrated to obtain

$$z(t) = z_0 + v_0 t - \frac{g}{2(1 + \frac{r^2}{k^2})}t^2 \tag{9}$$

Problem 2

H+F Chapter 6 Problem 12b

Since I_j doesn't contain time explicitly and the Lagrangian is invariant under the transformation that generates it, we know $\frac{dI_j}{dt} = \frac{\partial I_j}{\partial t} = 0$, so by the text's eqn (6.130) we have $[H, I_j] = 0$.

Problem 3

Prove that the Poisson Brackest satisfy the Jacobi identity.

This problem is a straightforward computational verification so we omit it's solution.

Problem 4

H+F Chapter 6 Problem 14

a)We have, using the easily verified formulae $[AB, C] = A[B, C]+[B, C]A,\ [A, BC] = [A, B]C + B[A, C]$,

$$[H, F] = [\frac{p^2}{2m}, x - \frac{pt}{m}] = \frac{1}{2m}(p[p, x] + [p, x]p) = -\frac{p}{m} \tag{10}$$

while

$$\frac{\partial F}{\partial t} = -\frac{p}{m} \tag{11}$$

so

$$\frac{dF}{dt} = [F, H] + \frac{\partial F}{\partial t} = 0 \tag{12}$$

so F is constant.

b) If F,G are constants of the motion we have, using eqn (6.130) a few times (which in this case just reads $[H, F] = \frac{\partial F}{\partial t}$), the Jacobi identity, and the antisymmetry of the Poisson Bracket,

$$\frac{d[F, G]}{dt} = [H, [F, G]] + \frac{\partial [F, G]}{\partial t} \tag{13}$$

$$= -[F, [G, H]] - [G, [H, F]] + [\frac{\partial F}{\partial t}, G] + [F, \frac{\partial G}{\partial t}] \tag{14}$$

$$
\begin{aligned}
&= -[F,[G,H]] - [G,[H,F]] - [[F,H],G] - [F,[G,H]] && (15)\\
&= 0 && (16)
\end{aligned}
$$

c)From (6.130),

$$\frac{\partial F}{\partial t} = \frac{dF}{dt} - [F,H] = -[F,H] \tag{17}$$

which is a constant of the motion by part b) and the fact that F,H were themselves constants of the motion by assumption.

Problem 5

H+F, Chapter 6 Problem 15

a) We'll prove $[l_x, l_y] = l_z$. The calculation for cyclic permutations is totally analogous. We make extensive use of the equation $[x_i, p_j] = \delta_{ij}$,the bilinearity of the Poisson Bracket (i.e. [A+B,C]=[A,C]+[B,C]), and the commutator product rule quoted in the solution to Problem 4a):

$$
\begin{aligned}
[l_x, l_y] &= [yp_z - zp_y, zp_x - xp_z] && (18)\\
&= [yp_z, zp_x] + [zp_y, xp_z] && (19)\\
&= y[p_z, z]p_x + x[z, p_z]p_y && (20)\\
&= xp_y - yp_x && (21)\\
&= l_z && (22)
\end{aligned}
$$

b) We'll compute the commutator of l_z with each component of $\mathbf{r}$ and $\mathbf{p}$, and then just quote the general result for the commutators with l_y and l_x.

$$
\begin{aligned}
[l_z, x] = [xp_y - yp_x, x] = -y[p_x, x] = y && (23)\\
[l_z, y] = [xp_y - yp_x, y] = x[p_y, y] = -x && (24)\\
[l_z, z] = [xp_y - yp_x, z] = 0 && (25)\\
[l_z, p_x] = [xp_y - yp_x, p_x] = [x, p_x]p_y = p_y && (26)\\
[l_z, p_y] = [xp_y - yp_x, p_y] = -[y, P_y]p_x = -p_x && (27)\\
[l_z, p_z] = [xp_y - yp_x, p_z] = 0 && (28)
\end{aligned}
$$

In general,

$$[l_i, r_j] = \epsilon_{ijk} r_k \tag{29}$$

$$[l_i, p_j] = \epsilon_{ijk} p_k \tag{30}$$