HW 8 SOLUTIONS

Problem 1

HF Chapter 6, problem 3

We are given a point transformation $Z(z, p, t) = Z(z, t) = z - D(t)$ and are asked to find the generating function $F_2(z, P, t)$ which yields this transformation. But to find F_2, we need to know the rest of the transformation, i.e. $P(z, p, t)$! Since we have a point transformation (i.e. Z has no p-dependence), the function $P(z, p, t)$ is automatically determined in the natural way: rewrite the Lagrangian in terms of Z and $\dot{Z}$, and then $P = \frac{\partial L}{\partial \dot{Z}}(Z(z, t), \dot{Z}(z, p, t))$. We compute explicitly:

$$L(z, \dot{z}) = \frac{m}{2}\dot{z}^2 - V(z) \tag{1}$$

so

$$L(Z, \dot{Z}) = \frac{m}{2}(\dot{Z} + \dot{D})^2 - V(Z + D) \tag{2}$$

so

$$P = \frac{\partial L}{\partial \dot{Z}} = m(\dot{Z} + \dot{D}) = m\dot{z} = p \tag{3}$$

so our PDE's for F_2 are

$$\frac{\partial F_2}{\partial P} = Z = z - D(t) \tag{4}$$

$$\frac{\partial F_2}{\partial z} = p \tag{5}$$

We can immediately integrate (4) to obtain

$$F_2 = zP - PD(t) + f(z, t) \tag{6}$$

where f is unknown. Plugging this into (5) yields

$$p = P + \partial_z f \tag{7}$$

so in view of (3) we can safely take $f \equiv 0$. Thus

$$F_2(z, P, t) = (z - D(t))P \tag{8}$$

The new Hamiltonian is

$$\bar{H}(Z,P,t) = H(z(Z,t),p(Z,P,t)) + \frac{\partial F_2}{\partial t} = \frac{P^2}{2m} + V(Z+D) - P\dot{D} \tag{9}$$

so Hamilton's equations are

$$\frac{\partial \bar{H}}{\partial P} = \dot{Z} = \frac{P}{m} - \dot{D} \tag{10}$$

$$-\frac{\partial \bar{H}}{\partial Z} = \dot{P} = -\frac{dV}{dz}(Z + D(t)) \tag{11}$$

Now, we can try and Legendre transform F_2 to get an F_1-type generating function, but

$$F_1(z,Z,t) = F_2(z,P,t) - PZ = (z-D)P - PZ = 0 \tag{12}$$

by the definition of Z.

The reason we cannot Legendre transform is that F_2 is not convex in P. This is the case whenever we have a point transformation $Q = Q(q,t)$; $\frac{\partial F_2}{\partial P} = Q(q)$ implies $F_2 = PQ(q) + f(q,t)$ which is not convex in P. Another way of looking at this is that $F_1(q,Q)$ cannot be a function on phase space because, for a point transformation, q and Q are not independent.

Problem 2

H+F Chapter 6 Problem 4

We have, since $F_4(p,P)$ is time independent,

$$P(z,p) = \bar{H}(Q(z,p),P(z,p)) = H(Q(z,p),P(z,p)) = \frac{p^2}{2m} + mgz \tag{13}$$

so our PDE's for F_4 are

$$\frac{\partial F_4}{\partial P} = Q \tag{14}$$

$$\frac{\partial F_4}{\partial p} = -z = \frac{1}{mg}(\frac{p^2}{2m} - P) \tag{15}$$

where we solved (13) for z and substituted in (15). We can integrate (15) to get

$$F_4(p,P) = -\frac{1}{mg}(pP - p^3/6m) \tag{16}$$

where we set to zero any function of P we might have added to F_4. This then yields

$$Q = \frac{\partial F_4}{\partial P} = -p/mg. \tag{17}$$

To show that Q is the time, note that

$$\dot{Q} = \frac{\partial H}{\partial P} = \frac{\partial P}{\partial P} = 1 \Rightarrow Q = t + c. \tag{18}$$

Problem 3

H+F Chapter 6 Problem 5

First we note that the determinant in question is just the Poisson bracket of Q and P:

$$\left|\frac{\partial(Q,P)}{\partial(q,p)}\right| = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = [Q,P]. \tag{19}$$

Now, in order to evaluate this Poisson bracket in terms of our generating function F(q,Q) we must be very careful in taking partial derivatives and keeping track of what set of variables we are using when we take a particular derivative. For instance, all the derivatives in $\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}$ are taken with the understanding that p and q are the independent variables, so we must express P in terms of p and q, i.e.

$$P = P(p,q) = \frac{\partial F}{\partial Q}(q, Q(q,p)). \tag{20}$$

Thus, using the chain rule,

$$\begin{aligned} \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} &= \frac{\partial Q}{\partial q}\frac{\partial^2 F}{\partial Q^2}\frac{\partial Q}{\partial p} - \left(\frac{\partial^2 F}{\partial Q \partial q} + \frac{\partial^2 F}{\partial Q^2}\frac{\partial Q}{\partial q}\right)\frac{\partial Q}{\partial p} \qquad (21) \\ &= \frac{\partial p}{\partial Q}|_{q,Q}\frac{\partial Q}{\partial p}|_{q,p} \end{aligned}$$

$$\tag{23}$$

where we used the chain rule in the first equality, $\frac{\partial F}{\partial q}|_{q,Q} = p$ in the second, all derivatives of F are taken with respect to the variables (q,Q), and we were explicit in the last line about the independent variables with respect to which the partial derivatives are taken. Now, recall that

$$p = p(q,p) = -\frac{\partial F}{\partial q}|_{q,Q}(q, Q(q,p)) \tag{24}$$

so by the chain rule

$$1 = \frac{\partial p}{\partial p}|_{q,p} = \frac{\partial p}{\partial Q}|_{q,Q}\frac{\partial Q}{\partial p}|_{q,p} \tag{25}$$

so by (22) we have

$$\left|\frac{\partial(Q,P)}{\partial(q,p)}\right| = [Q,P] = 1 \tag{26}$$

Problem 4

H+F chapter 6 Problem 10

Not a one-liner

a)To check if $Q(q,p) = ln(\frac{sinp}{q})$, $P(q,p) = qcotp$ is a canonical transformation we compute the Poisson Bracket of Q and P:

$$\begin{aligned}
[Q,P] &= \frac{\partial}{\partial q}ln(\frac{sinp}{q})\frac{\partial}{\partial p}(qcotp) - \frac{\partial}{\partial p}(ln(\frac{sinp}{q}))\frac{\partial}{\partial q}(qcotp) && (27)\\
&= \frac{1}{q}qcsc^2(p) - cot^2(p) && (28)\\
&= 1 && (29)
\end{aligned}$$

so this transformation is canonical.

b) We compute

$$\begin{aligned}
p\,dq - P\,dQ &= p\,dq - qcotp(-\frac{1}{q}\,dq + cotp\,dp) && (30)\\
&= (p + cotp)dq + q(1 - csc^2(p))dp && (31)\\
&= d(pq + qcotp) && (32)
\end{aligned}$$

c)From $Q(q,p) = ln(\frac{sinp}{q})$ we have $p = sin^{-1}(qe^Q) = \frac{\partial F_1}{\partial q}$ so integrating yields

$$F_1(q,Q) = \sqrt{e^{-2Q} - q^2} + qsin^{-1}(qe^Q) + f(Q) \tag{33}$$

where f(Q) is an arbitrary function of Q. Differentiating $F_1(q,Q)$ with respect to Q then yields (after some manipulation)

$$\frac{\partial F_1}{\partial Q} = f'(Q) - e^{-Q}\sqrt{1-q^2e^{2Q}} \tag{34}$$

but we must have, using $sinp = qe^Q$ and hence $cotp = \frac{1}{qe^Q}\sqrt{1-q^2e^{2Q}} = \sqrt{\frac{e^{-2Q}}{q^2}-1}$,

$$\frac{\partial F_1}{\partial Q} = -P = -qcotp = -e^{-Q}\sqrt{1-q^2e^{2Q}} \tag{35}$$

so compatibility with (34) requires $f(Q) =$ constant which we set to 0. Thus

$$F_1(q,Q) = \sqrt{e^{-2Q}-q^2} + qsin^{-1}(qe^Q). \tag{36}$$

Alternatively, one can note that $p\,dq - P\,dQ = \frac{\partial F_1}{\partial q}\,dq - \frac{\partial F_1}{\partial Q}\,dQ = dF$ so that $F_1 = p(q,Q)q + qcot(p(q,Q))$. Substituting in our expression for $p(q,Q)$ yields (36).

Problem 5

H+F chapter 6 Problem 8

Another one-liner:

$$[F,G]_{Q,P} = \left|\frac{\partial(F,G)}{\partial(Q,P)}\right| = \left|\frac{\partial(F,G)}{\partial(q,p)}\right|\left|\frac{\partial(q,p)}{\partial(Q,P)}\right| = [F,G]_{q,p} \tag{37}$$

since

$$\left|\frac{\partial(q,p)}{\partial(Q,P)}\right| = [q,p]_{Q,P} = 1. \tag{38}$$