S-Duality and Chern-Simons Theory

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String Theory Seminar

University of Texas at Austin

Based on

- Yoon Pyo Hong and OG, "S-duality and Chern-Simons Theory," [arXiv:hep-th/0812.1213]
- Yoon Pyo Hong and OG, "S-twisted compactification of $N = 4$, Topological 2+1D Quantum Field Theory, and Minimal Strings" [arXiv:hep-th/0902.????]

S-duality

$$\tau \equiv \frac{4\pi i}{g_{\rm YM}^2} + \frac{\theta}{2\pi}$$

$$\mathbf{s} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} \in {\rm SL}(2,\mathbb{Z})$$

$$\tau \to \frac{\mathbf{a}\tau + \mathbf{b}}{\mathbf{c}\tau + \mathbf{d}}\,.$$

S-duality's action on states

Temporal gauge: $\qquad A_0 = 0.$

$$\widetilde{\Psi}(A) \equiv \int [\mathcal{D}\widetilde{A}]\mathcal{S}(A,\widetilde{A})\Psi(\widetilde{A})$$

$$\tau \to \frac{\mathbf{a}\tau + \mathbf{b}}{\mathbf{c}\tau + \mathbf{d}}\,, \qquad E_i \to \mathbf{a}E_i + \mathbf{b}B_i\,, \quad B_i \to \mathbf{c}E_i + \mathbf{d}B_i\,.$$

[Lozano; Gaiotto & Witten]

$$\mathcal{S}(A,\widetilde{A}) = \exp\left\{\frac{i}{4\pi\mathbf{c}}\int(\mathbf{d}A\wedge dA - 2\widetilde{A}\wedge dA + \mathbf{a}\widetilde{A}\wedge d\widetilde{A})\right\}.$$

$$\widetilde{E}_i\mathcal{S} = \mathcal{S}(\mathbf{a}E_i + \mathbf{b}B_i), \qquad \widetilde{B}_i\mathcal{S} = \mathcal{S}(\mathbf{c}E_i + \mathbf{d}B_i).$$

$$E_i \equiv -2\pi i\delta/\delta A_i$$

$U(1)$ Chern-Simons from S-duality

$$\widetilde{\Psi}\{A\} \equiv \int [\mathcal{D}\widetilde{A}]\mathcal{S}(A,\widetilde{A})\Psi(\widetilde{A})$$

$$\mathcal{S}(A,\widetilde{A}) = \exp\left\{\frac{i}{4\pi\mathbf{c}}\int(\mathbf{d}A\wedge dA - 2\widetilde{A}\wedge dA + \mathbf{a}\widetilde{A}\wedge d\widetilde{A})\right\}.$$

$$A = \widetilde{A} \Longrightarrow \mathcal{I}(A) \equiv \frac{\mathbf{a}+\mathbf{d}-2}{4\pi\mathbf{c}}\int A\wedge dA\,.$$

CS level: $\qquad k \equiv (\mathbf{a}+\mathbf{d}-2)/\mathbf{c}.$

Physical interpretation?

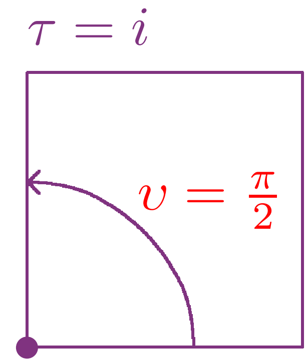
Selfduality

$$\tau = \frac{\mathbf{a}\tau + \mathbf{b}}{\mathbf{c}\tau + \mathbf{d}} \Longrightarrow \mathbf{c}\tau + \mathbf{d} = e^{iv}.$$

At a selfdual τ we can compactify on a circle with an S-twist.

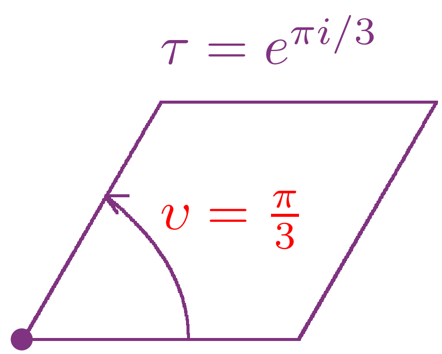
$$\mathbf{s} \equiv \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\tau \longrightarrow -\frac{1}{\tau} \qquad |k| = \left|\frac{\mathbf{a}+\mathbf{d}-2}{\mathbf{c}}\right| = 2$$



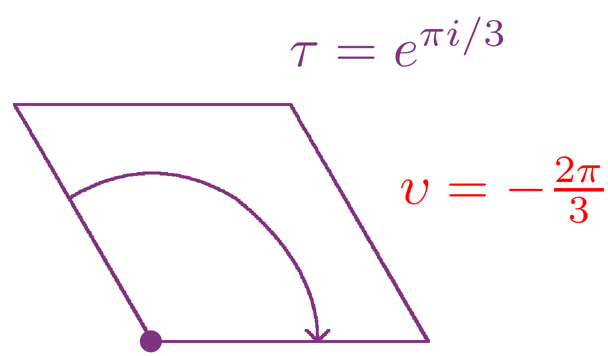
$$\mathbf{s} \equiv \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\tau \longrightarrow \frac{\tau-1}{\tau} \qquad |k| = \left|\frac{\mathbf{a}+\mathbf{d}-2}{\mathbf{c}}\right| = 1$$

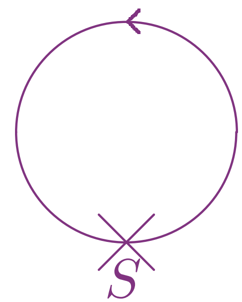


$$\mathbf{s} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\tau \longrightarrow \frac{\tau-1}{\tau} \qquad |k| = 3$$



periodic time



$N = 4$ Super Yang-Mills

A_μ	gauge field	$\mu = 0 \ldots 3$
Φ^I	adjoint-valued scalars	$I = 1 \ldots 6$
ψ^a_α	adjoint-valued spinors	$a = 1 \ldots 4$ and $\alpha = 1, 2$
$\overline{\psi}_{a\dot{\alpha}}$	complex conjugate spinors	$a = 1 \ldots 4$ and $\dot{\alpha} = \dot{1}, \dot{2}$
$Q_{a\alpha}$	SUSY generators	$a = 1 \ldots 4$ and $\alpha = 1, 2$
$\overline{Q}^a_{\dot{\alpha}}$	complex conjugate generators	$a = 1 \ldots 4$ and $\dot{\alpha} = \dot{1}, \dot{2}$

$$Z^1 = \Phi^1 + i\Phi^4, \qquad Z^2 = \Phi^2 + i\Phi^5, \qquad Z^3 = \Phi^3 + i\Phi^6.$$

Supersymmetry

$$\mathbf{s} : \tau \rightarrow \frac{\mathbf{a}\tau + \mathbf{b}}{\mathbf{c}\tau + \mathbf{d}}$$

$$\mathbf{s} : Q_{a\alpha} \rightarrow \left(\frac{\mathbf{c}\tau + \mathbf{d}}{|\mathbf{c}\tau + \mathbf{d}|} \right)^{1/2} Q_{a\alpha} = e^{\frac{i\upsilon}{2}} Q_{a\alpha}$$

[Kapustin & Witten]

$$\mathbf{s} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Longrightarrow \upsilon = \frac{\pi}{2}$$

R-Symmetry

$Spin(6) \simeq SU(4)$

$$\gamma \equiv \begin{pmatrix} e^{i\varphi_1} & & & \\ & e^{i\varphi_2} & & \\ & & e^{i\varphi_3} & \\ & & & e^{i\varphi_4} \end{pmatrix} \in SU(4)\,, \qquad \left(\sum_a \varphi_a = 0\right)\,,$$

acts as

$$\gamma(\psi^a_\alpha) = e^{i\varphi_a}\psi^a_\alpha\,, \qquad \gamma(\overline{\psi}_{a\alpha}) = e^{-i\varphi_a}\overline{\psi}_{a\alpha}\,, \qquad a = 1\dots4.$$

$$\gamma(Z^k) = e^{i(\varphi_k+\varphi_4)}Z^k\,, \qquad k = 1\dots3.$$

Combined R-S- action

$$Q_{a\alpha} \to e^{\frac{iv}{2} - i\varphi_a} Q_{a\alpha} \, .$$

$\Longrightarrow N = 2r$ invariant generators

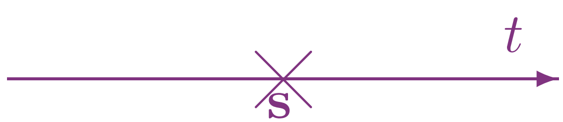
$r = \#\{a$ for which $e^{i\varphi_a} = e^{iv/2}\}$

R- and S- twisted boundary conditions

$$\Phi(x = 0^-) = \gamma[\Phi(x = 0^+)]$$

$$Z^k(x = 0^-) = e^{i(\varphi_k + \varphi_4)}\Phi(x = 0^+), \qquad k = 1, 2, 3$$

$\ldots$

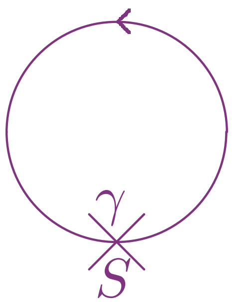


$$\Psi(A, \ldots)|_{t=0^+} = \int [\mathcal{D}\widetilde{A}]\mathcal{S}(A, \widetilde{A})\Psi(\widetilde{A}, \ldots)|_{t=0^-}$$

SUSY in 2+1D

$\Longrightarrow N = 2r, \qquad r = \#\{a \text{ for which } e^{i\varphi_a} = e^{iv/2}\}$

$$\gamma = \begin{pmatrix} e^{\frac{i}{2}v} & & & \\ & e^{\frac{i}{2}v} & & \\ & & e^{\frac{i}{2}v} & \\ & & & e^{-\frac{3i}{2}v} \end{pmatrix} \Longrightarrow N = 6$$



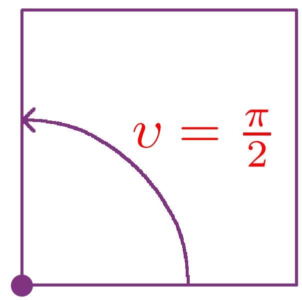
$$\gamma = \begin{pmatrix} e^{\frac{i}{2}v} & & & \\ & e^{\frac{i}{2}v} & & \\ & & e^{-i(v+\varphi_4)} & \\ & & & e^{i\varphi_4} \end{pmatrix} \Longrightarrow N = 4$$

$\gamma =$ R-symmetry twist $\qquad e^{iv} \equiv \mathbf{c}\tau + \mathbf{d}$

$$\mathbf{s} \equiv \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

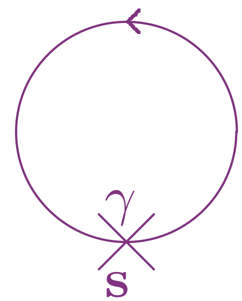
$$\tau \to -\frac{1}{\tau}$$

$\tau = i$



$\mathbf{s}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$
$\gamma(v)$

$N = 4$ SYM



$N = 6$
in 2+1D

IR???

$$\mathbf{s} \equiv \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

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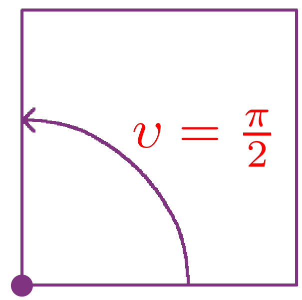
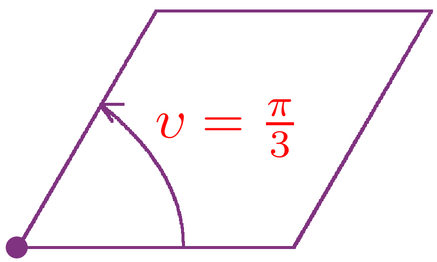


$$\mathbf{s} \equiv \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\tau \longrightarrow \frac{\tau - 1}{\tau}$$

$\tau = e^{\pi i/3}$



$$\mathbf{s} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

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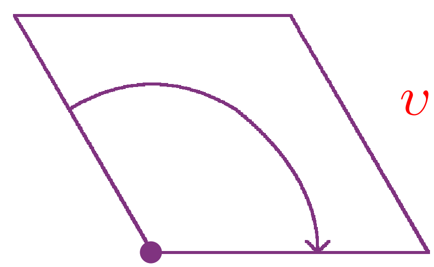
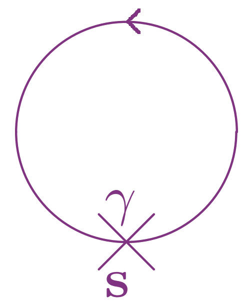


$\upsilon = -\frac{2\pi}{3}$

$\mathbf{s}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$
$\gamma(\upsilon)$

$N = 4$ SYM

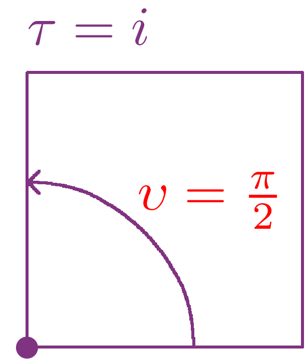


$N = 6$
in 2+1D

IR???

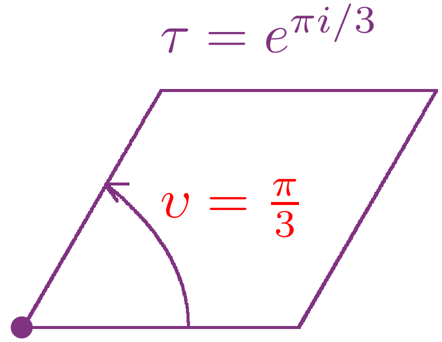
$$\mathbf{s} \equiv \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\tau \longrightarrow -\frac{1}{\tau}$ CS at $k = 2$?



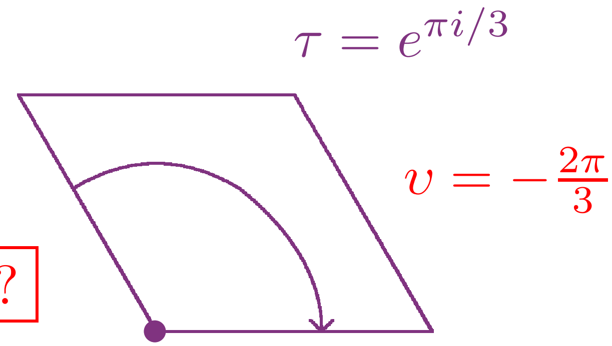
$$\mathbf{s} \equiv \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

$\tau \longrightarrow \frac{\tau - 1}{\tau}$ CS at $k = 1$?



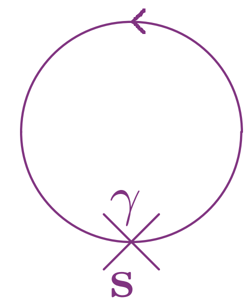
$$\mathbf{s} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

$\tau \longrightarrow \frac{\tau - 1}{\tau}$ CS at $k = 3$?



$\mathbf{s}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$
$\gamma(\upsilon)$

$N = 4$ SYM



$N = 6$
in 2+1D

IR???

Moduli

$$Z \equiv Z^1 \equiv \phi^1 + i\phi^4$$

BPS operators:

$$\mathcal{O}_p \equiv g_{\mathrm{YM}}^{-p}\,\mathrm{tr}(Z^p)\,, \qquad p = 1, 2, \ldots$$

These operators are $\mathrm{SL}(2,\mathbb{Z})$-duality invariant [Intriligator].

Action of R-symmetry twist:

$$(\mathcal{O}_p)^\gamma = e^{ip\upsilon}\mathcal{O}_p\,.$$

$\mathcal{O}_p$ is single-valued if and only if $e^{ip\upsilon} = 1$.

Moduli . . .

- for $\tau = i$ and $\mathbf{s} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, $\langle \mathcal{O}_p \rangle \neq 0$ requires $p \in 4\mathbb{Z}$;

- for $\tau = e^{\pi i/3}$ and $\mathbf{s} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ $\langle \mathcal{O}_p \rangle \neq 0$ requires $p \in 6\mathbb{Z}$;

- for $\tau = e^{\pi i/3}$ and $\mathbf{s} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ $\langle \mathcal{O}_p \rangle \neq 0$ requires $p \in 3\mathbb{Z}$.

For $U(n)$, $\mathcal{O}_{n+1}, \mathcal{O}_{n+2}, \ldots$ are not independent of $\mathcal{O}_1, \ldots, \mathcal{O}_n$. Thus for $\tau = i$ and $\mathbf{s} = \mathbf{s}'$, for example, if $n < 4$ none of the operators $\mathcal{O}_p$ can get a VEV.

States on T^2 from String Theory

type	brane	1	2	3	4	5	6	7	8	9	10	
IIB	D3	$=$	$=$	$\times$							$\vert$	T on 1:
IIA	D2	$\circ$	$=$	$\times$							$\vert$	to M:
M	M2	$\circ$	$=$	$\times$							$\circ$	on 2:
IIA	F1	$\circ$	$\vert$	$\times$							$\circ$	

Legend:

- $\vert$ direction doesn't exist in the theory;
- $=$ a direction that the brane wraps;
- $\times$ a direction that the brane wraps and has the S-R-twist;
- $\circ$ a compact direction that the brane doesn't wrap;

Counting fixed-points

type	brane	1	2	3	4	5	6	7	8	9	10
IIB	D3	$=$	$=$	$\times$							$\vert$
IIA	F1	$\circ$	$\vert$	$\times$							$\circ$

$$\tau = i \Longrightarrow g_{\mathrm{IIB}} = 1 \Longrightarrow R_1 = R_{10}.$$

Directions $1, 10$ form a T^2 of complex structure τ;

F1-strings are n points in directions $1, 10$;

F1-strings are wound in direction 3;

Counting fixed-points ...

Directions $1, 10$ form a T^2 of complex structure τ;

F1-strings are n points in directions $1, 10$;

F1-strings are wound in direction 3;

S-R-twist is entirely geometrical!

It is a rotation by $\upsilon = \pi/2$ of T^2;

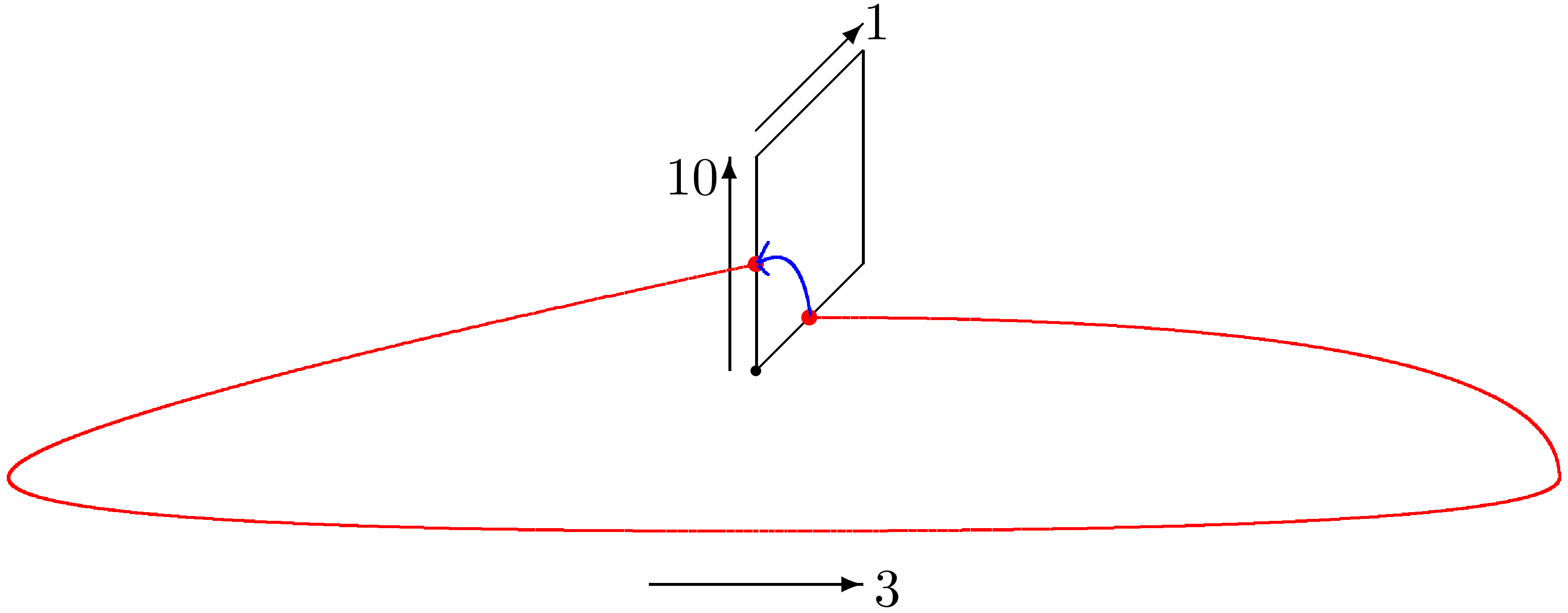
Need to find fixed points of this rotation (up to S_n);

$\{z_{\sigma(1)}, \ldots, z_{\sigma(n)}\} = \{z_1, \ldots, z_n\}$ up to $\mathbb{Z} + \mathbb{Z}\tau$;

One Ramond-Ramond ground state for each fixed point.

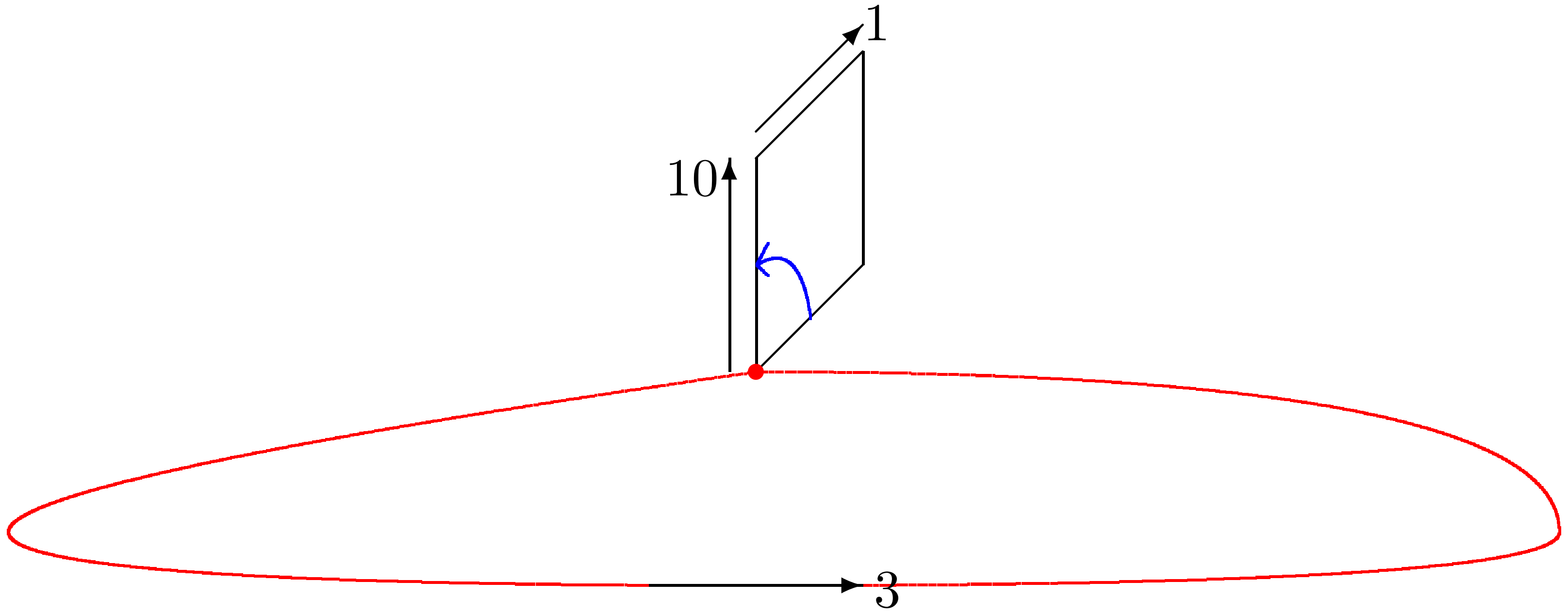
T^2 (directions $1, 10$) fibered over S^1 (direction 3):

Geometrical twist and wound string



T^2 (directions $1, 10$) fibered over S^1 (direction 3):

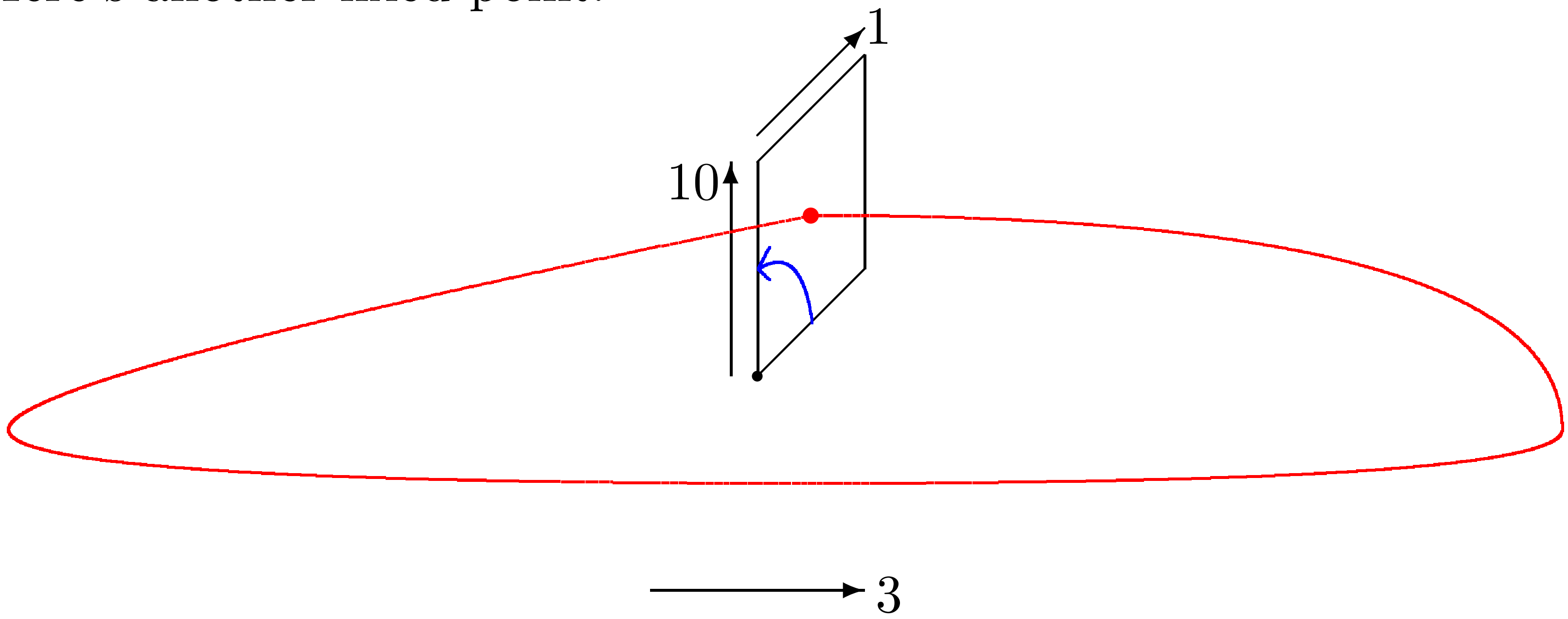
Geometrical twist and wound string
Minimal energy configuration: find fixed points of twist!



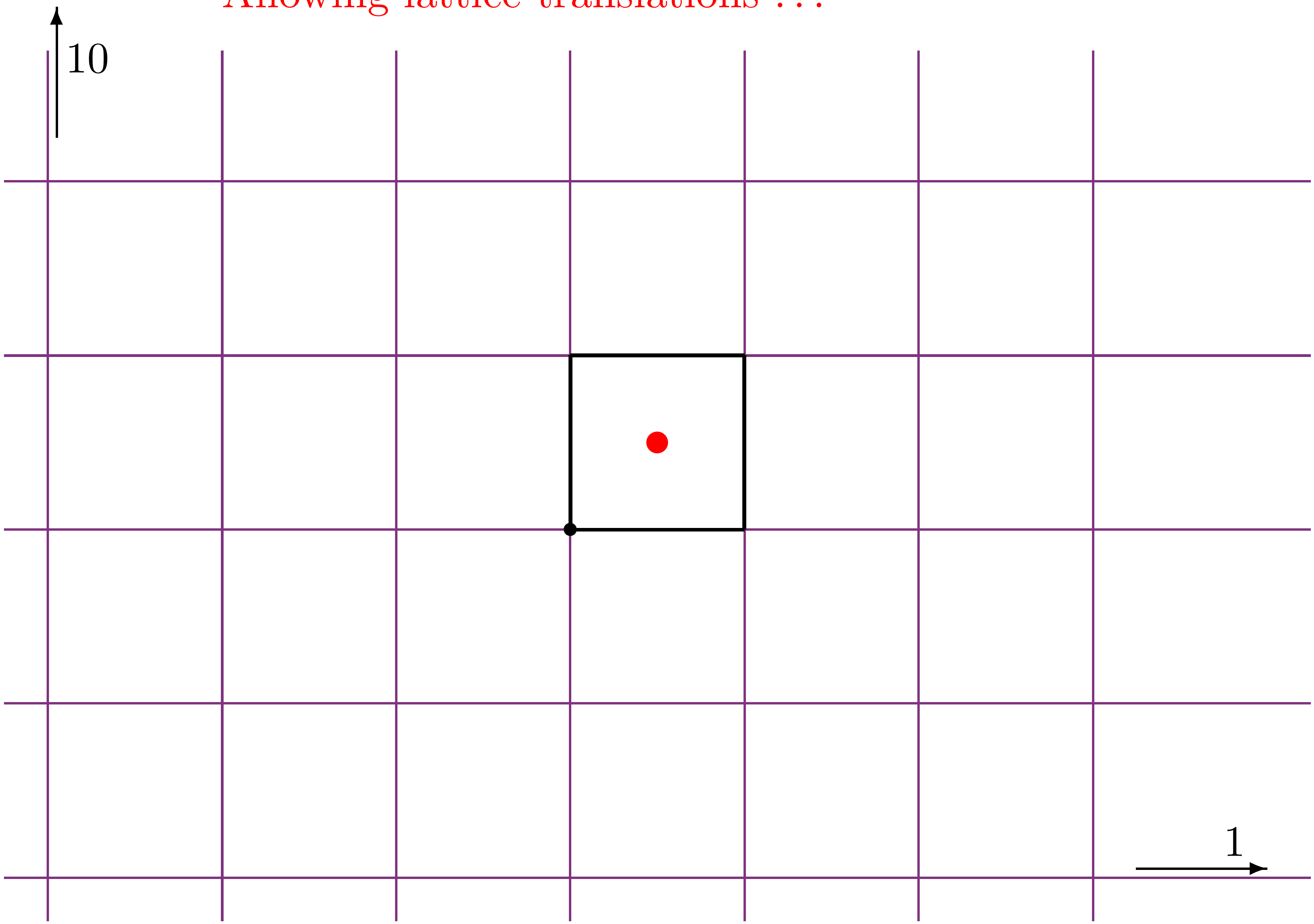
T^2 (directions $1, 10$) fibered over S^1 (direction 3):

Geometrical twist and wound string
Minimal energy configuration: find fixed points of twist!

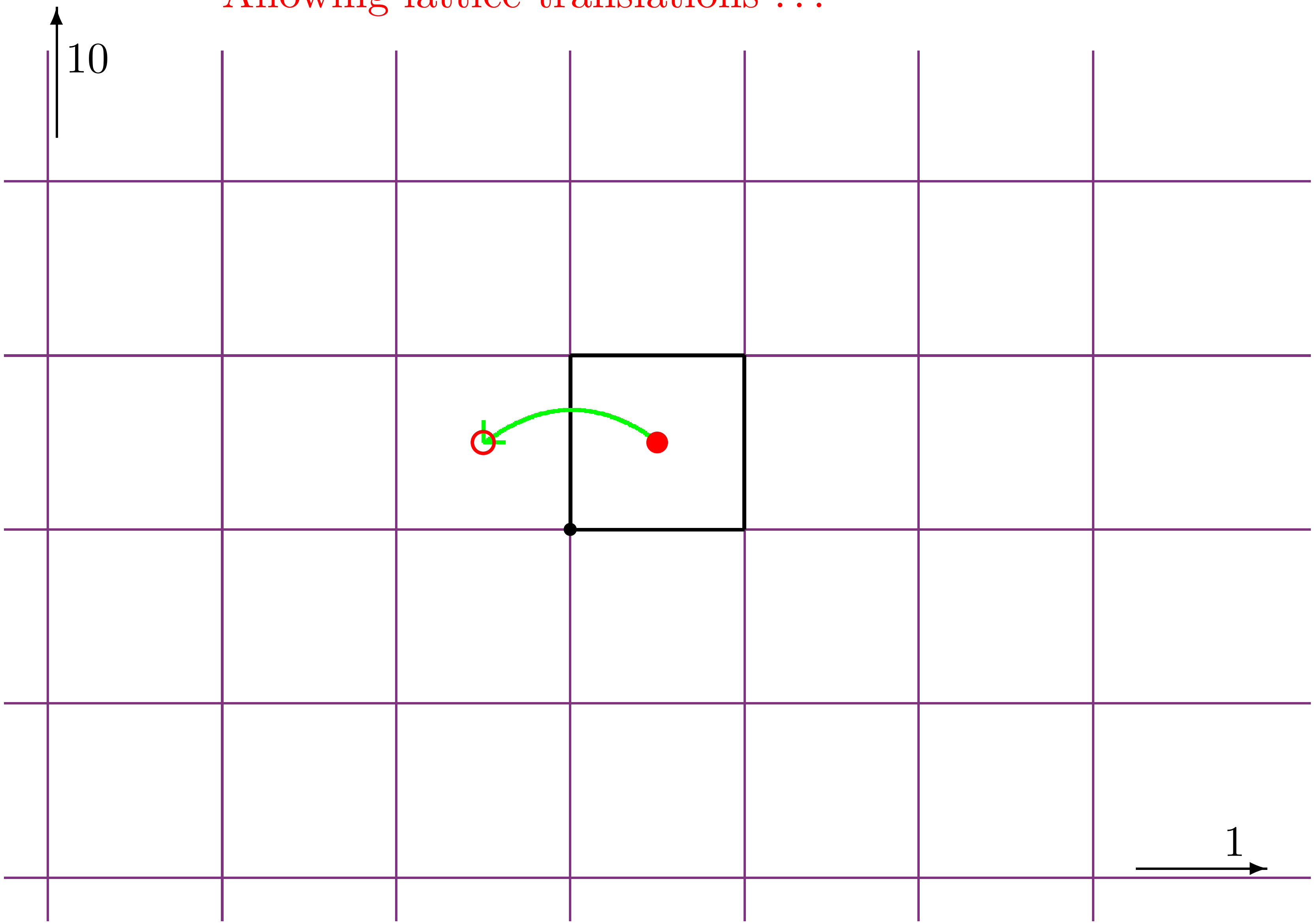
Here's another fixed point.

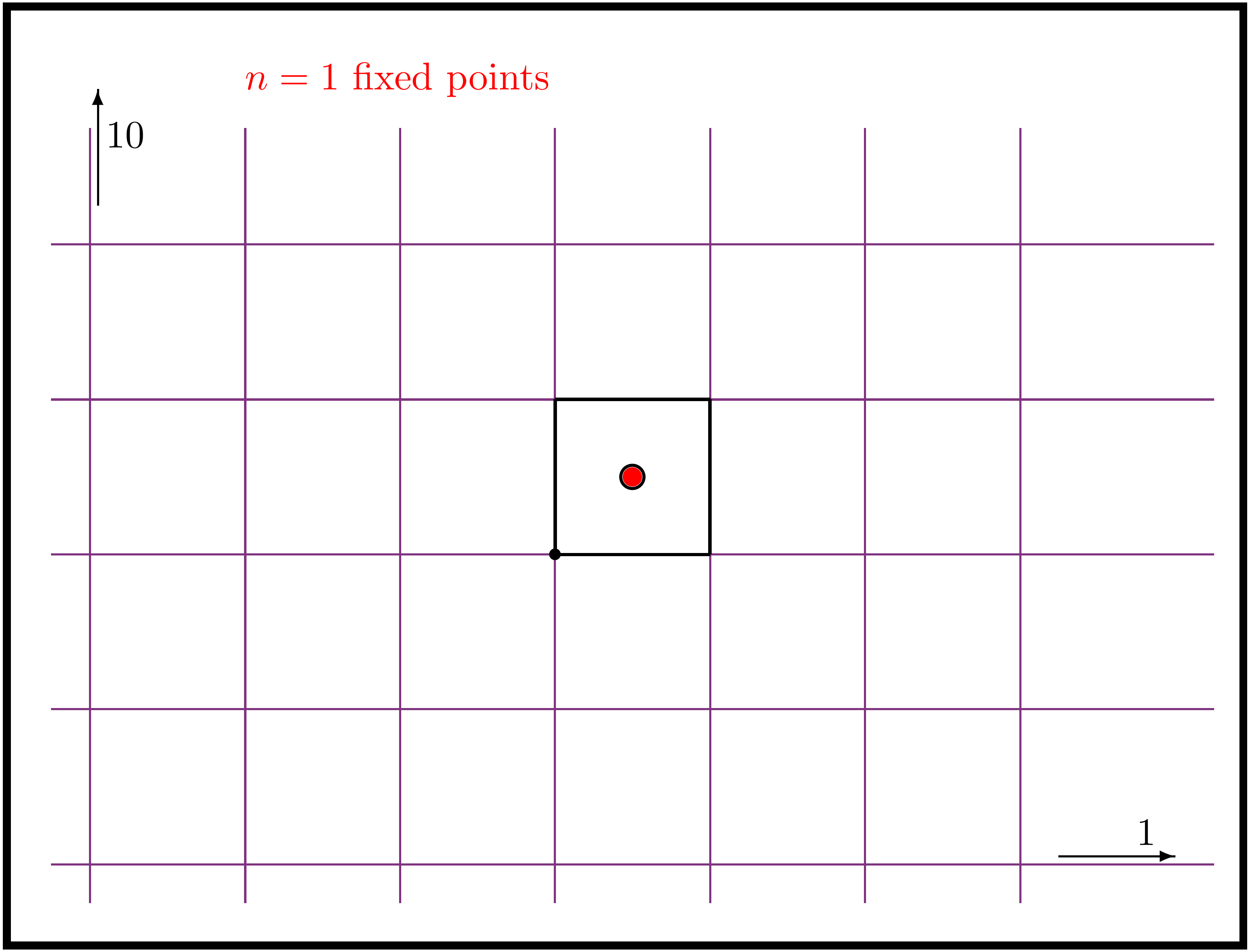


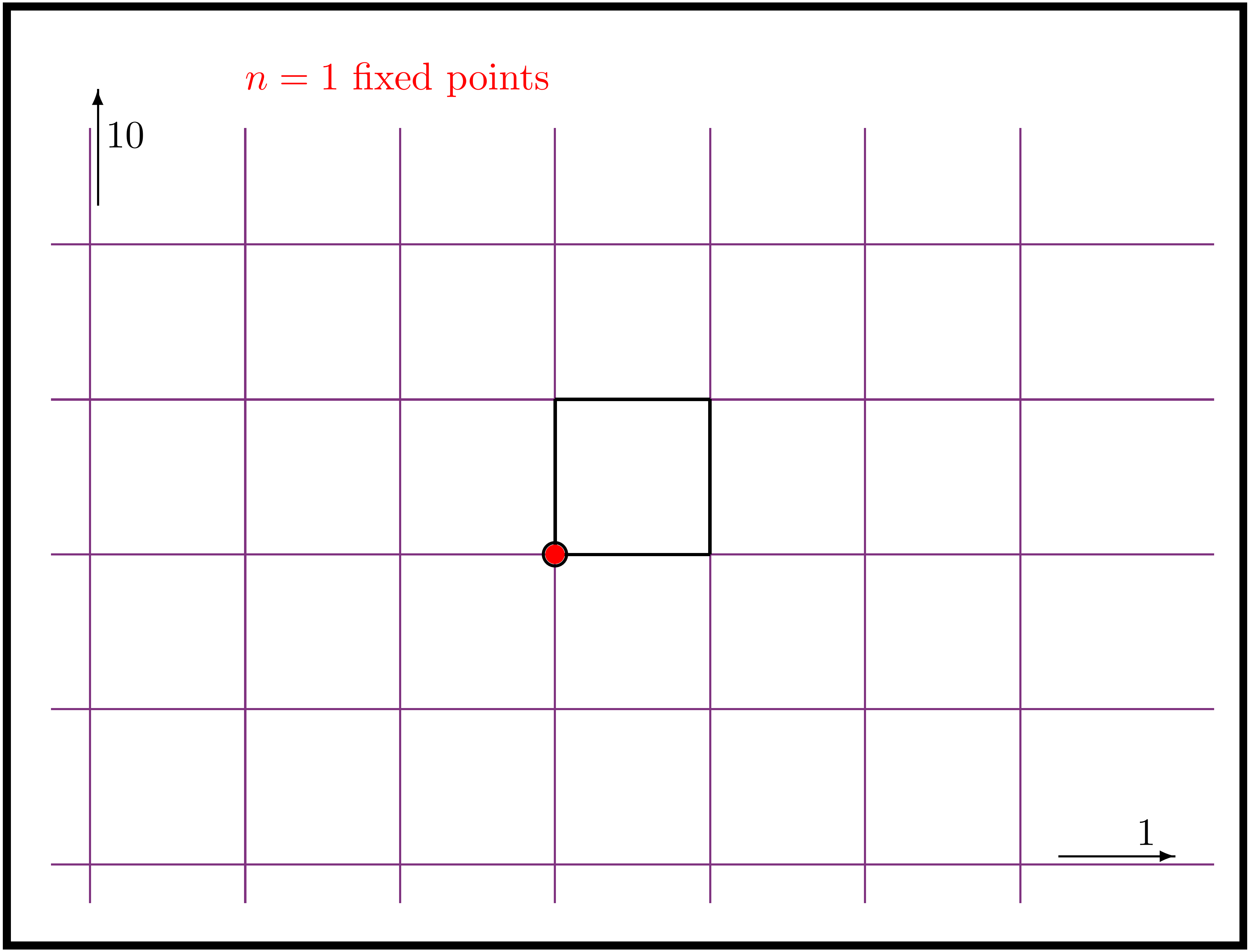
Allowing lattice translations . . .

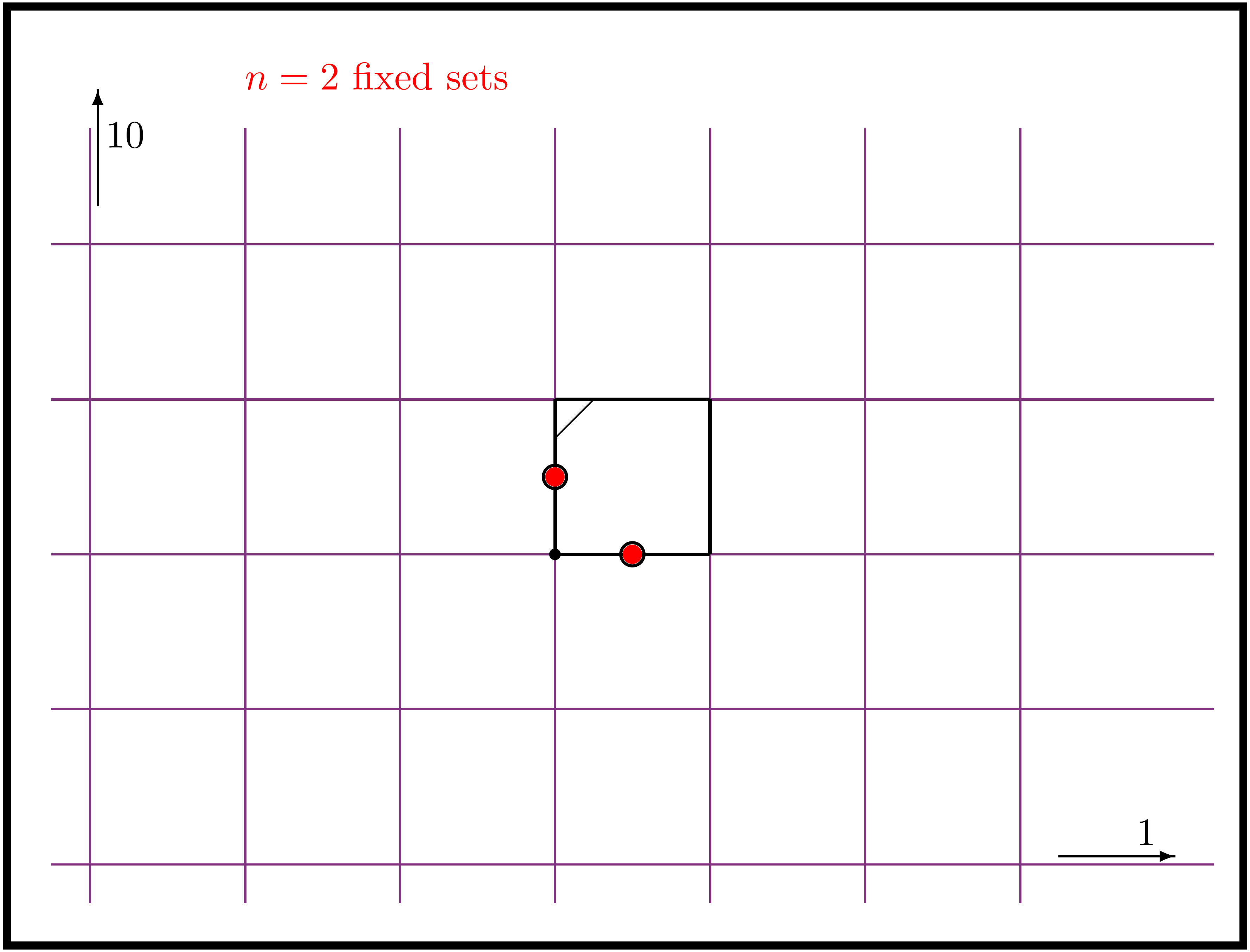


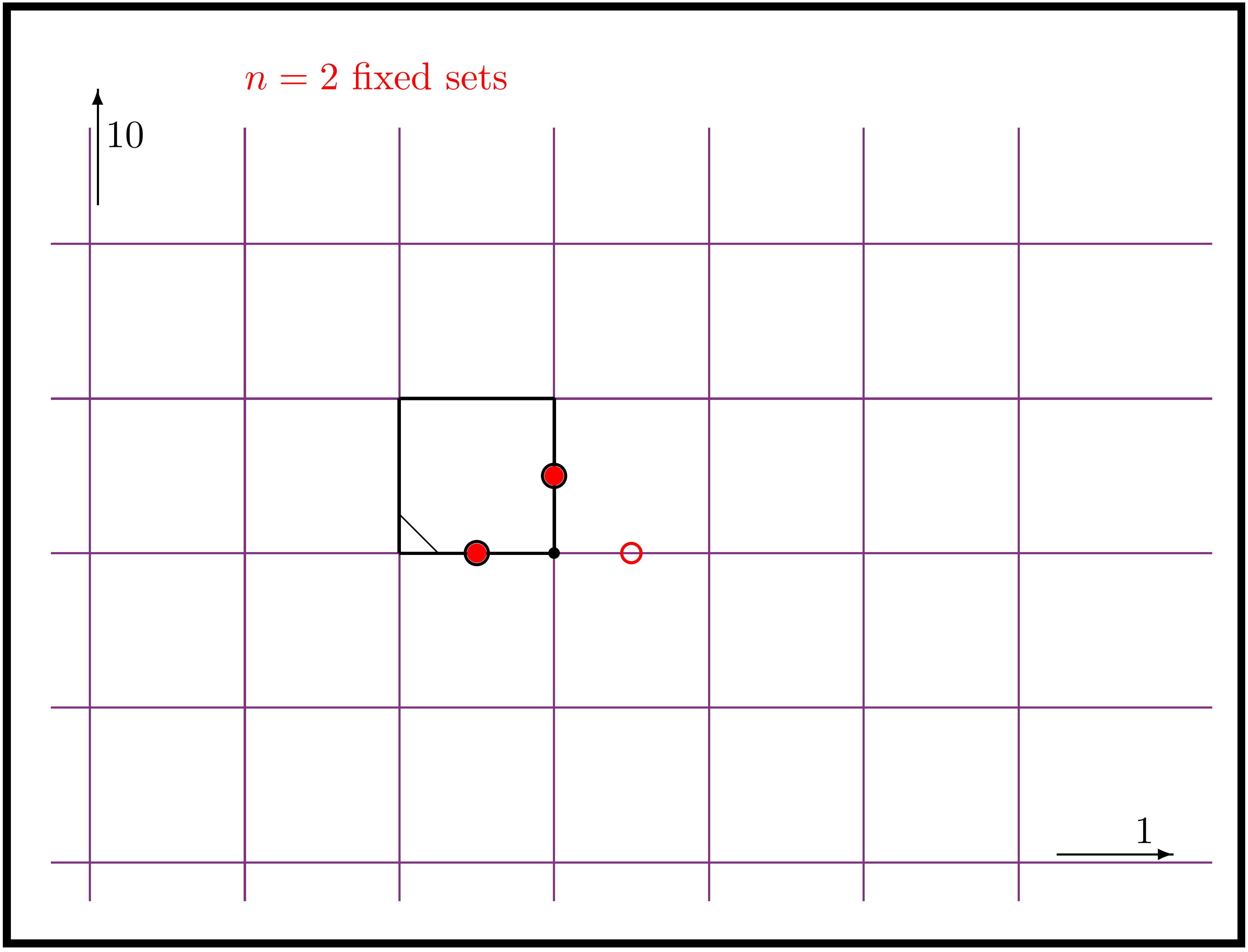
Allowing lattice translations . . .

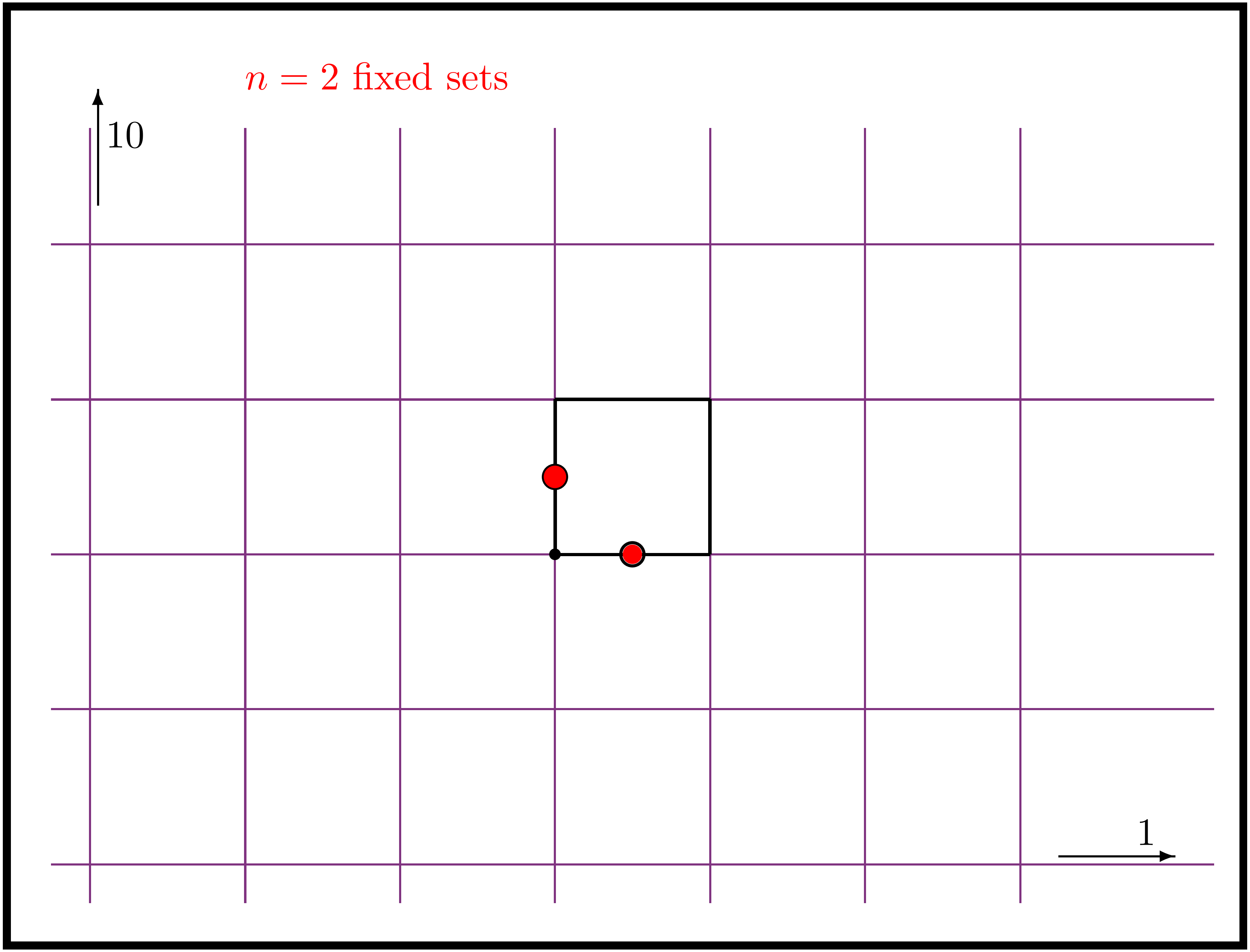


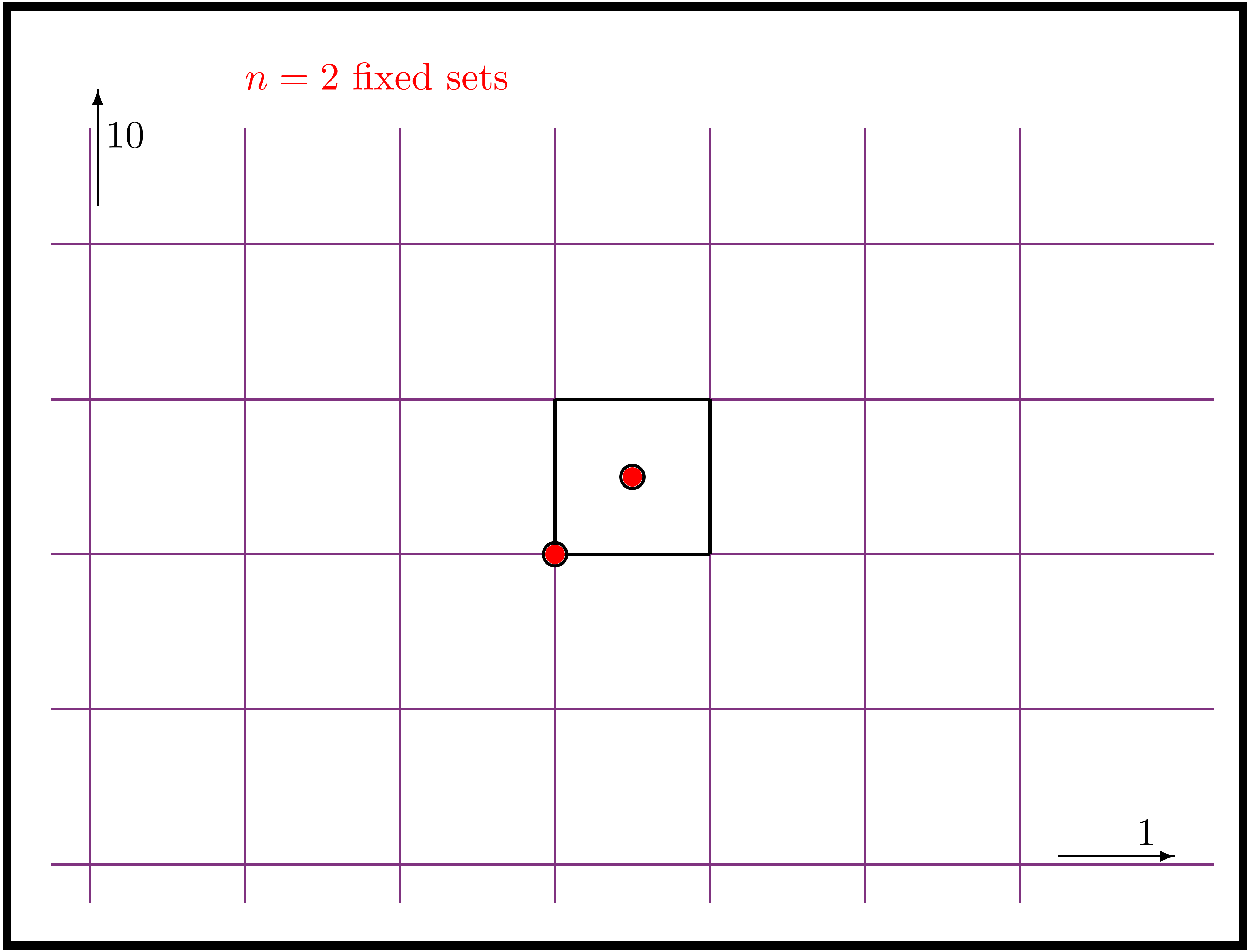

$n = 1$ fixed points
10
1


$n = 1$ fixed points
10
1

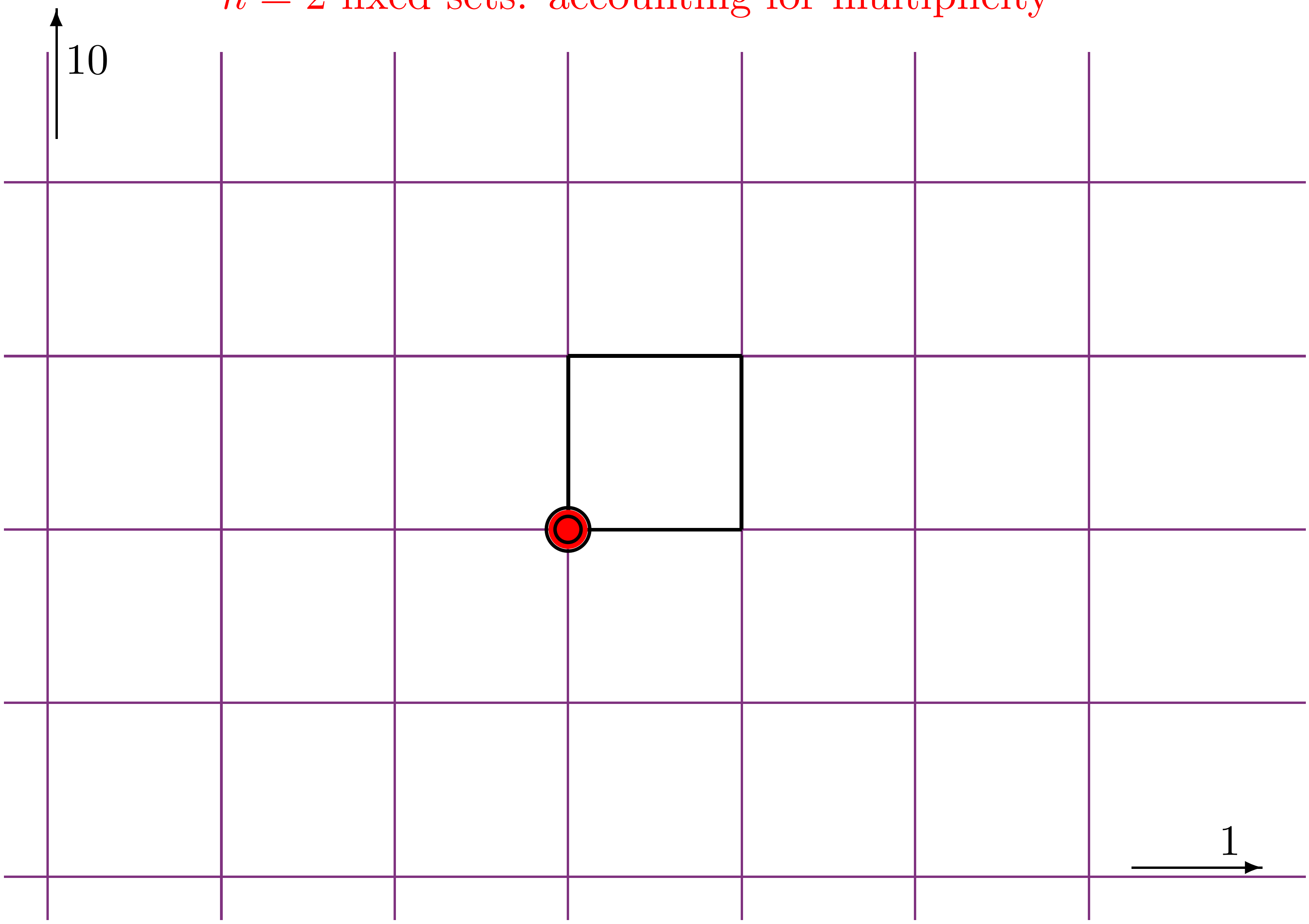

$n = 2$ fixed sets
10
1


$n = 2$ fixed sets
10
1


$n = 2$ fixed sets
10
1

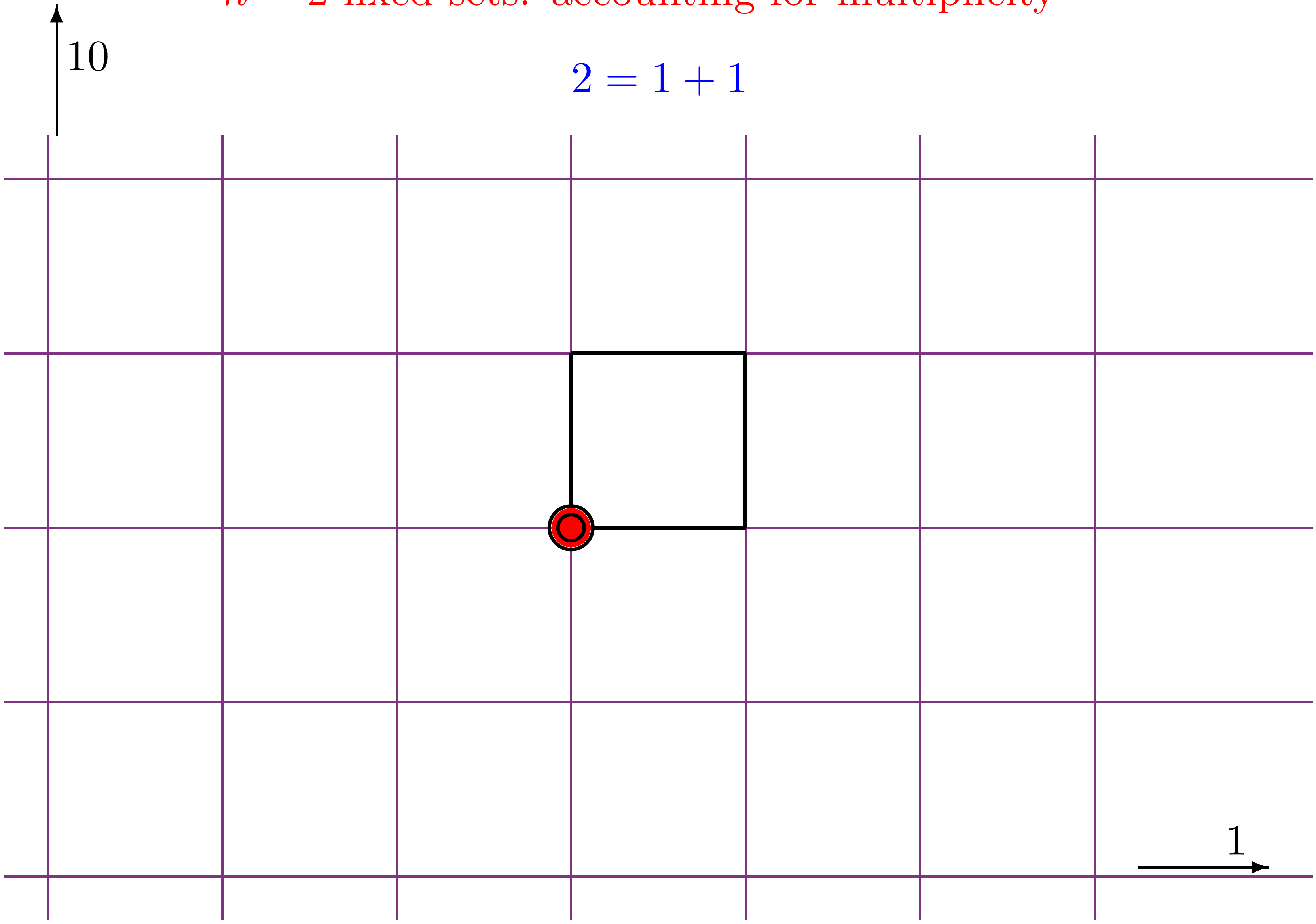

$n = 2$ fixed sets
10
1

$n = 2$ fixed sets: accounting for multiplicity



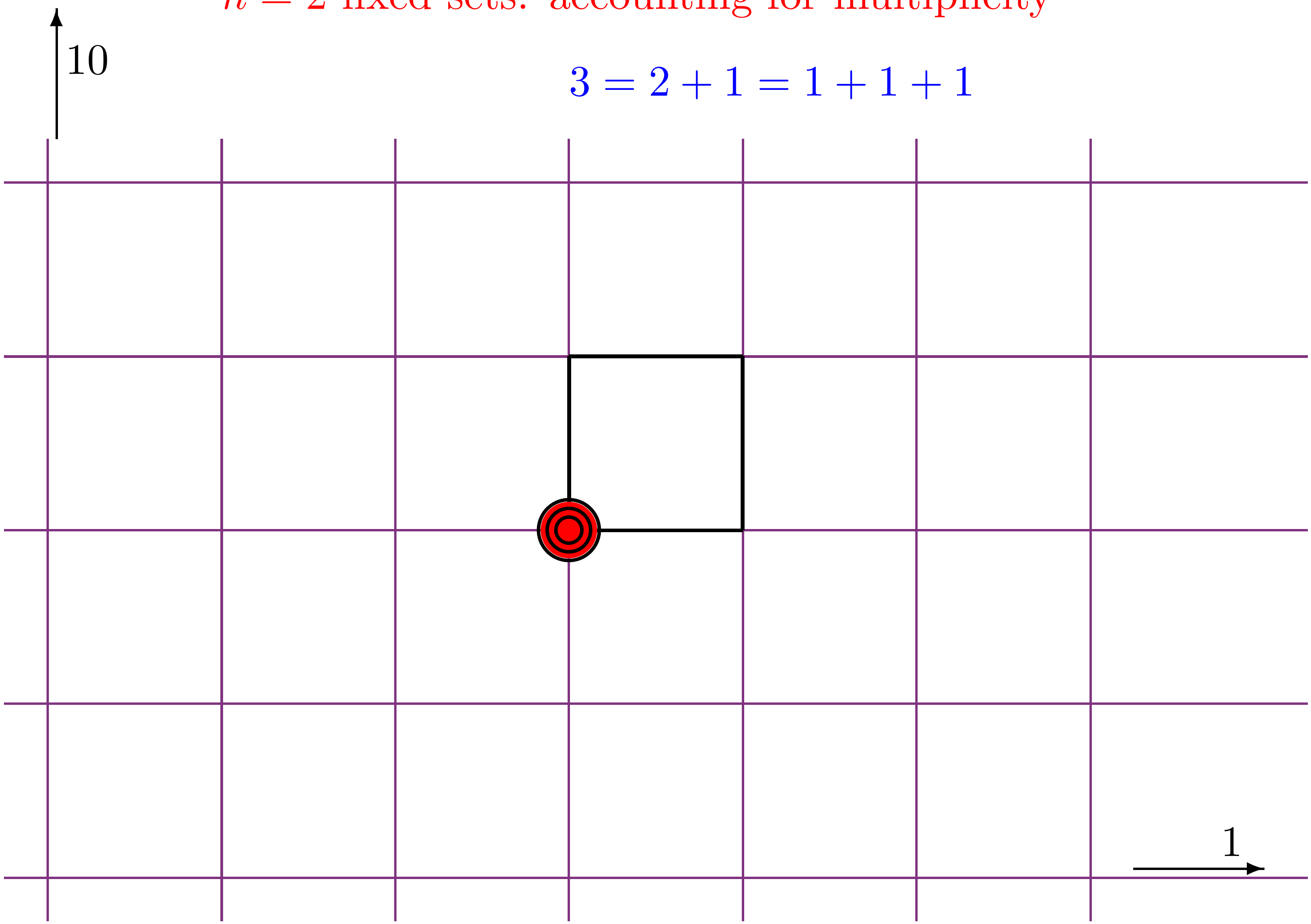
$n = 2$ fixed sets: accounting for multiplicity

$2 = 1 + 1$



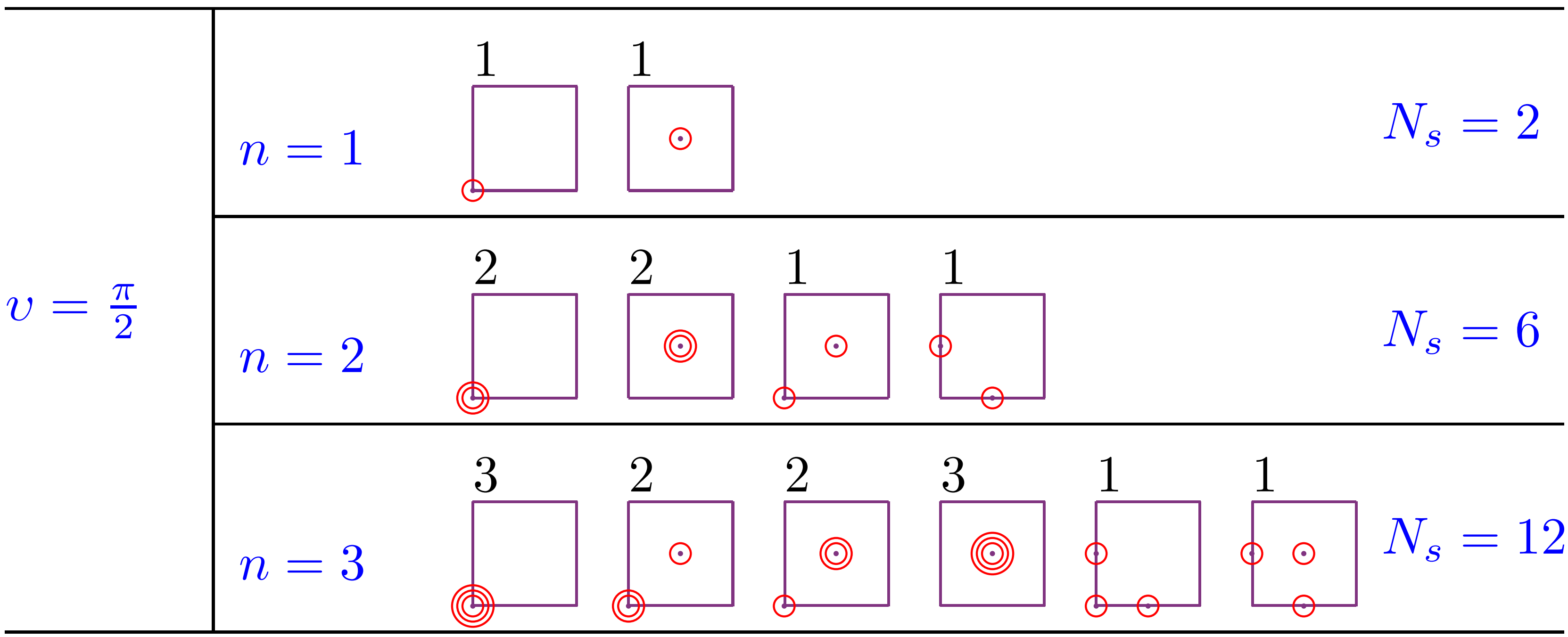
$n = 2$ fixed sets: accounting for multiplicity

$$3 = 2 + 1 = 1 + 1 + 1$$



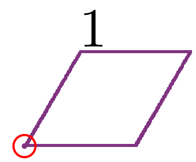
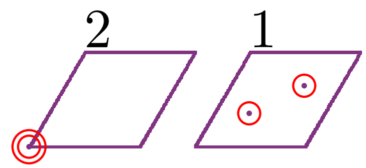
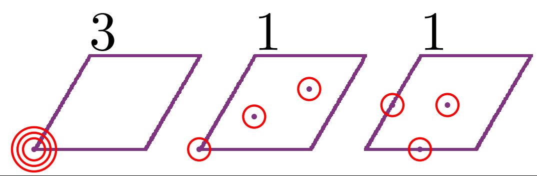
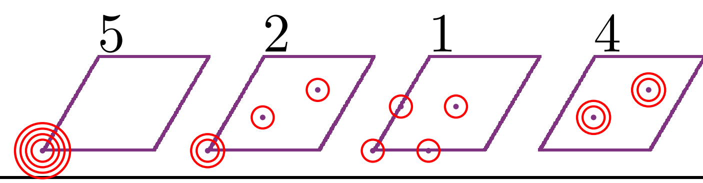
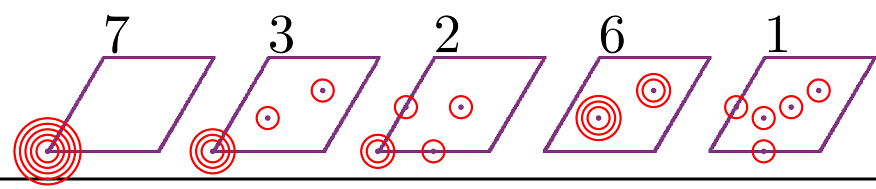
Counting number of ground states

(Singlet RR ground state)



Counting number of ground states

(Singlet RR ground state)

$v = \frac{\pi}{3}$	$n = 1$		$N_s = 1$
	$n = 2$		$N_s = 3$
	$n = 3$		$N_s = 5$
	$n = 4$		$N_s = 12$
	$n = 5$		$N_s = 19$

Number of states for $U(n)$ on T^2							
			n				
τ	υ	$\|k\|$	1	2	3	4	5
$e^{i\pi/3}$	$\frac{\pi}{3}$	1	**1**	**3**	**5**	**12**	**19**
i	$\frac{\pi}{2}$	2	**2**	**6**	**12**		
$e^{i\pi/3}$	$-\frac{2\pi}{3}$	3	**3**	**9**			

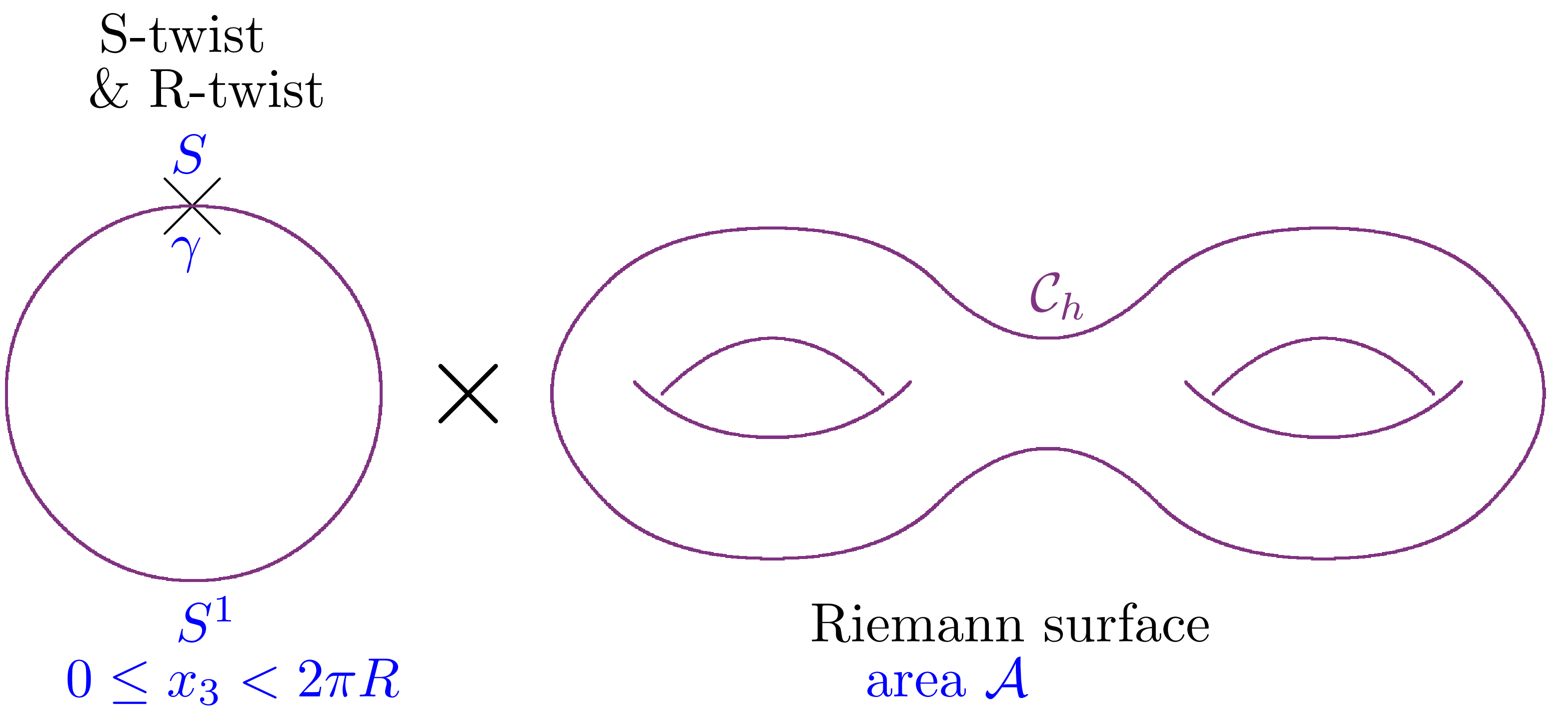
Number of states for $U(n)$ on T^2							
			n				
τ	υ	$\|k\|$	1	2	3	4	5
$e^{i\pi/3}$	$\frac{\pi}{3}$	1	**1**	**3**	**5**	**12**	**19**
i	$\frac{\pi}{2}$	2	**2**	**6**	**12**		
$e^{i\pi/3}$	$-\frac{2\pi}{3}$	3	**3**	**9**			

Chern-Simons:

$U(1)$ level k: $N_s = k$.

$SU(2)$ level k: $N_s = k + 1$.

$SU(3)$ level k: $N_s = (k+1)(k+2)/2$.



$\mathcal{A} \to 0 \Longrightarrow \sigma$-model on $\mathcal{M}_H$

S-duality becomes T-duality [Bershadsky & Johansen & Sadov & Vafa; Harvey & Moore & Strominger]

Witten Index

$$\#\{\text{vacua of 2+1D theory on } \mathcal{C}_h\} = I = \text{tr}_0\{(-1)^F \mathcal{T}(\mathbf{s})\gamma\}\,.$$

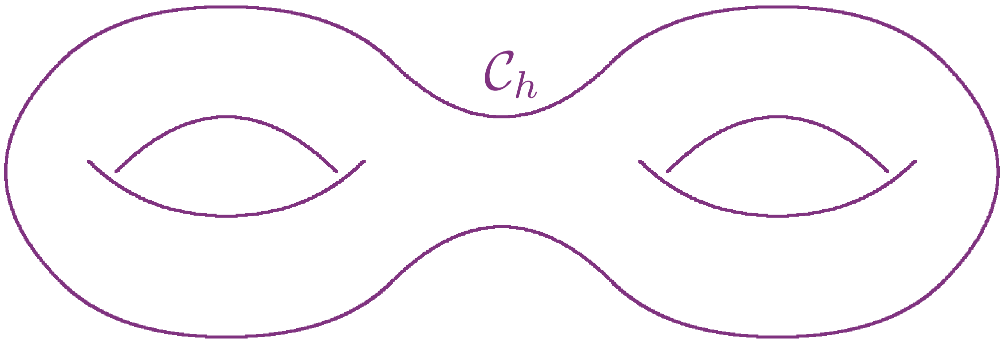
Hitchin's equations

$$F_{z\overline{z}} = [\phi_z, \overline{\phi}_{\overline{z}}]$$

$$D_z \overline{\phi}_{\overline{z}} = D_{\overline{z}} \phi_z = 0$$

A_z gauge field

ϕ_z adj.-valued 1-form



Riemann surface

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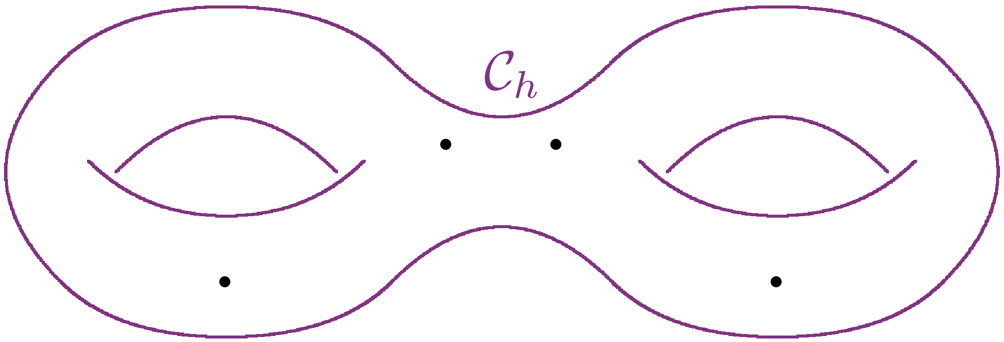
$b_{zz} = \mathrm{tr}(\phi_z^2)$ holomorphic with $4h - 4$ zeroes.

Space of quadratic differentials: $\mathbb{C}^{3(h-1)}$



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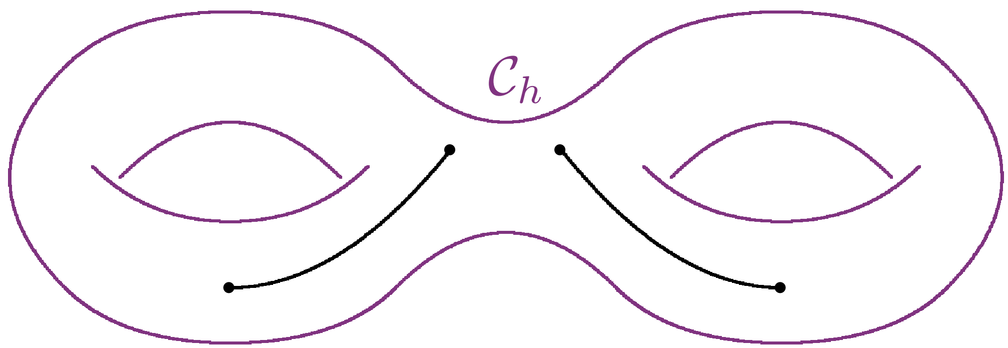
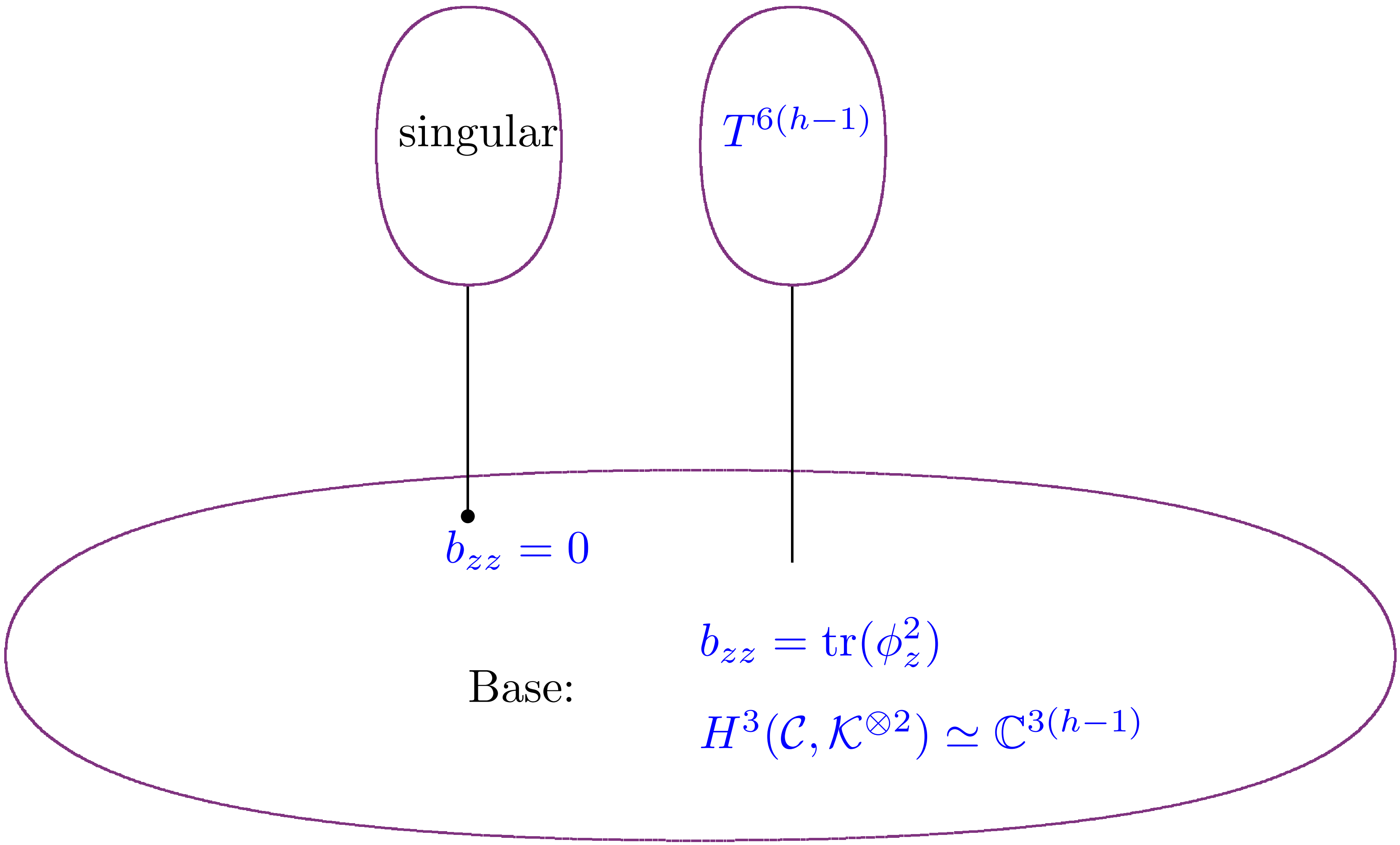


Riemann surface

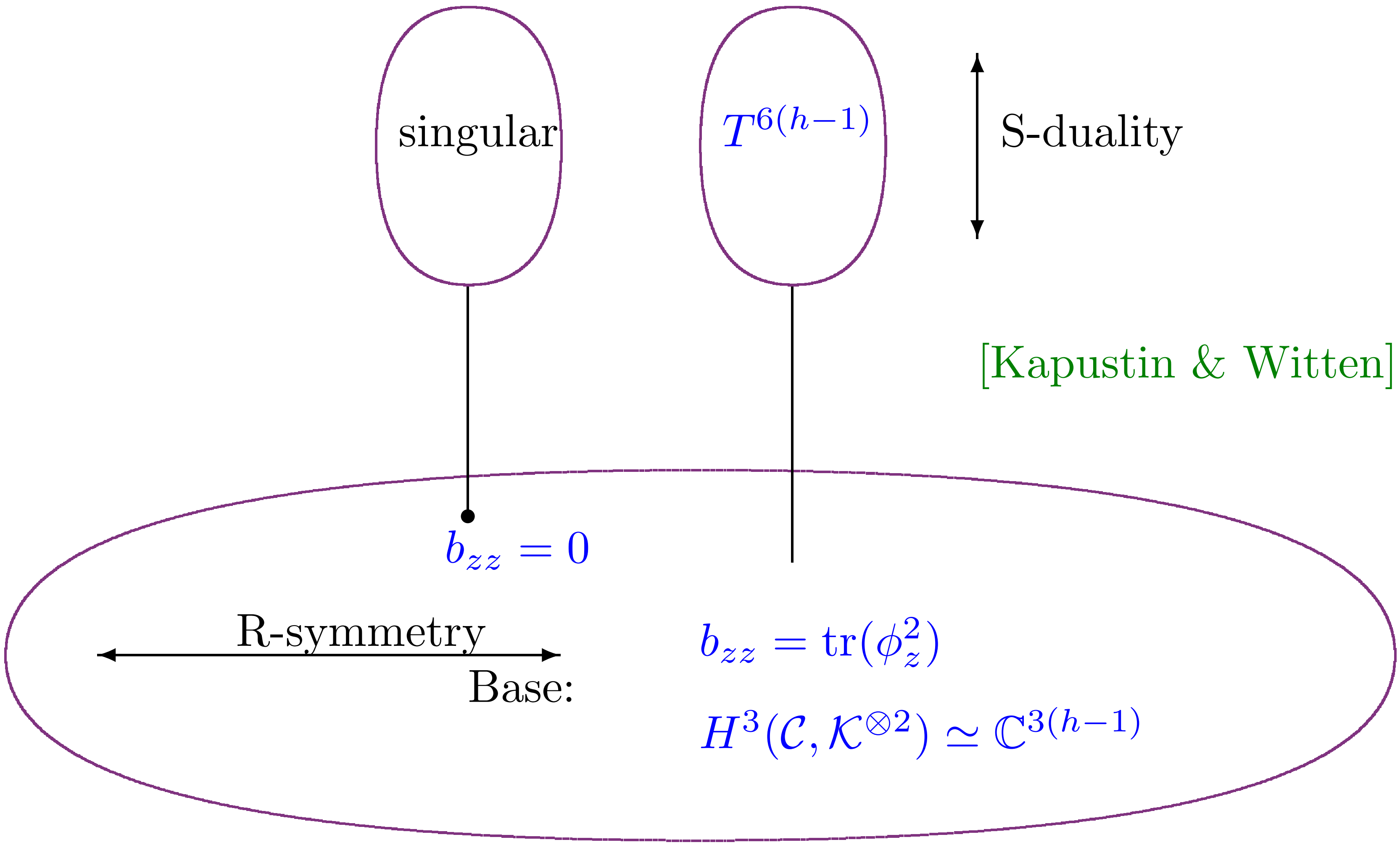
Double cover has genus $4h - 3$

Prym subspace of its Jacobian: $T^{6(h-1)}$

Hitchin Fibration



Hitchin Fibration



The fiber pver $b_{zz} = 0$ . . .

$$b_{zz} = \mathrm{tr}(\phi_z^2) = 0$$

Case 1: $\phi_z = 0 \Longrightarrow \mathcal{M}_{\mathrm{fc}}$ = moduli space of flat connections.

Case 2: $\phi_z = \begin{pmatrix} 0 & \alpha_z \\ 0 & 0 \end{pmatrix}, \qquad A_{\overline{z}} = \begin{pmatrix} a_{\overline{z}} & c_{\overline{z}} \\ 0 & -a_{\overline{z}} \end{pmatrix},$

$$a_{\overline{z}} = -\frac{1}{2}\partial_{\overline{z}} \log \alpha_z \,,$$

$$\partial_z a_{\overline{z}} - \partial_{\overline{z}} a_z = |\alpha_z|^2 + |c_{\overline{z}}|^2, \qquad \text{and } \frac{c^*_{\overline{z}}}{\alpha_z} = \text{holomorphic.}$$

Special subcase of 2: $c_{\overline{z}} = 0$.

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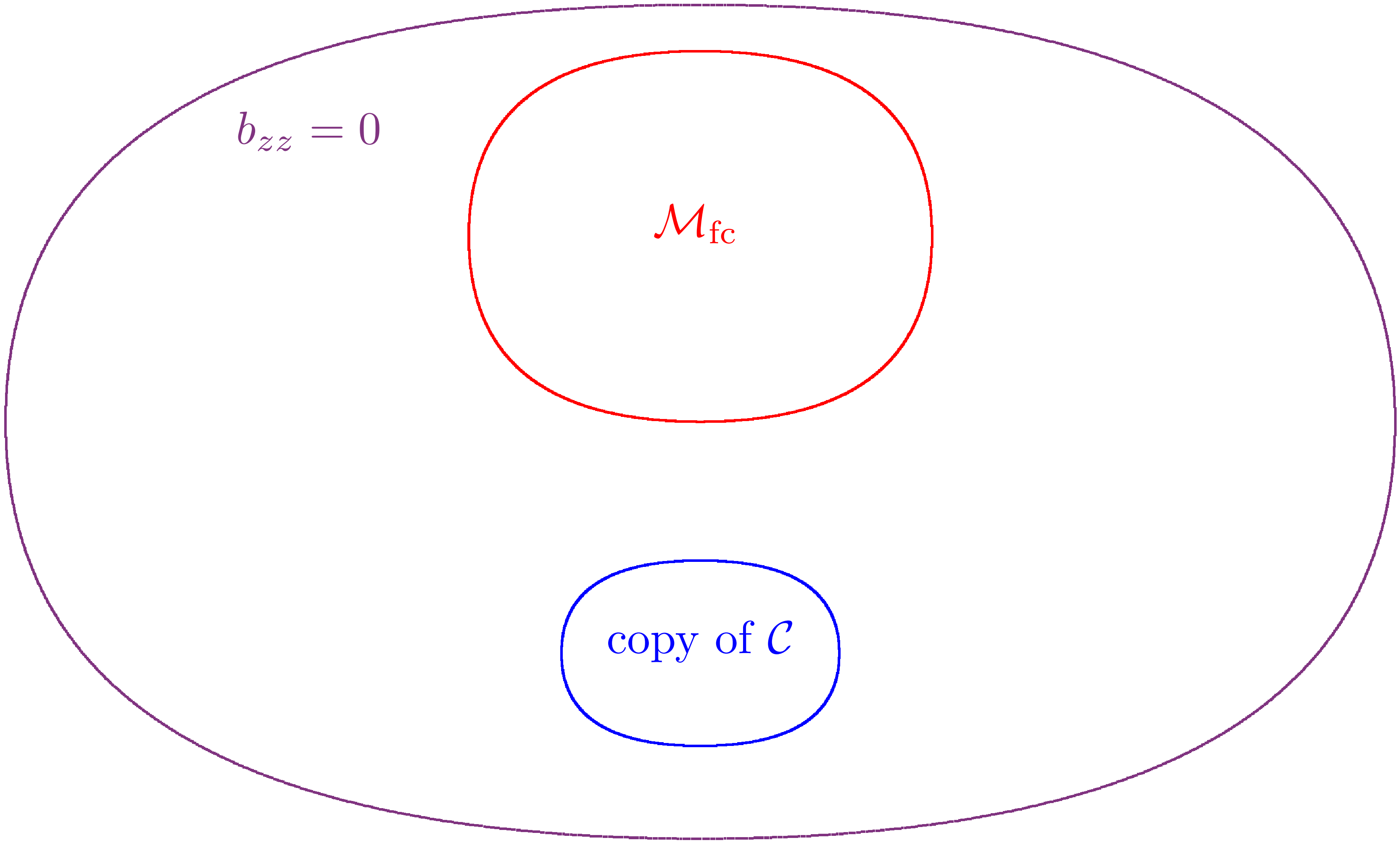
Special subcase of 2: $c_{\overline{z}} = 0$.

if also genus $h = 2$: α_z has a single simple zero on $\mathcal{C}_2$ which determines the solution uniquely up to gauge.

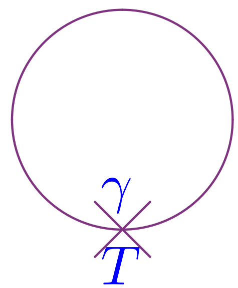
The fiber over $b_{zz} = 0$ . . .



T-duality and Geometric Quantization

1+1D σ-model with target space X

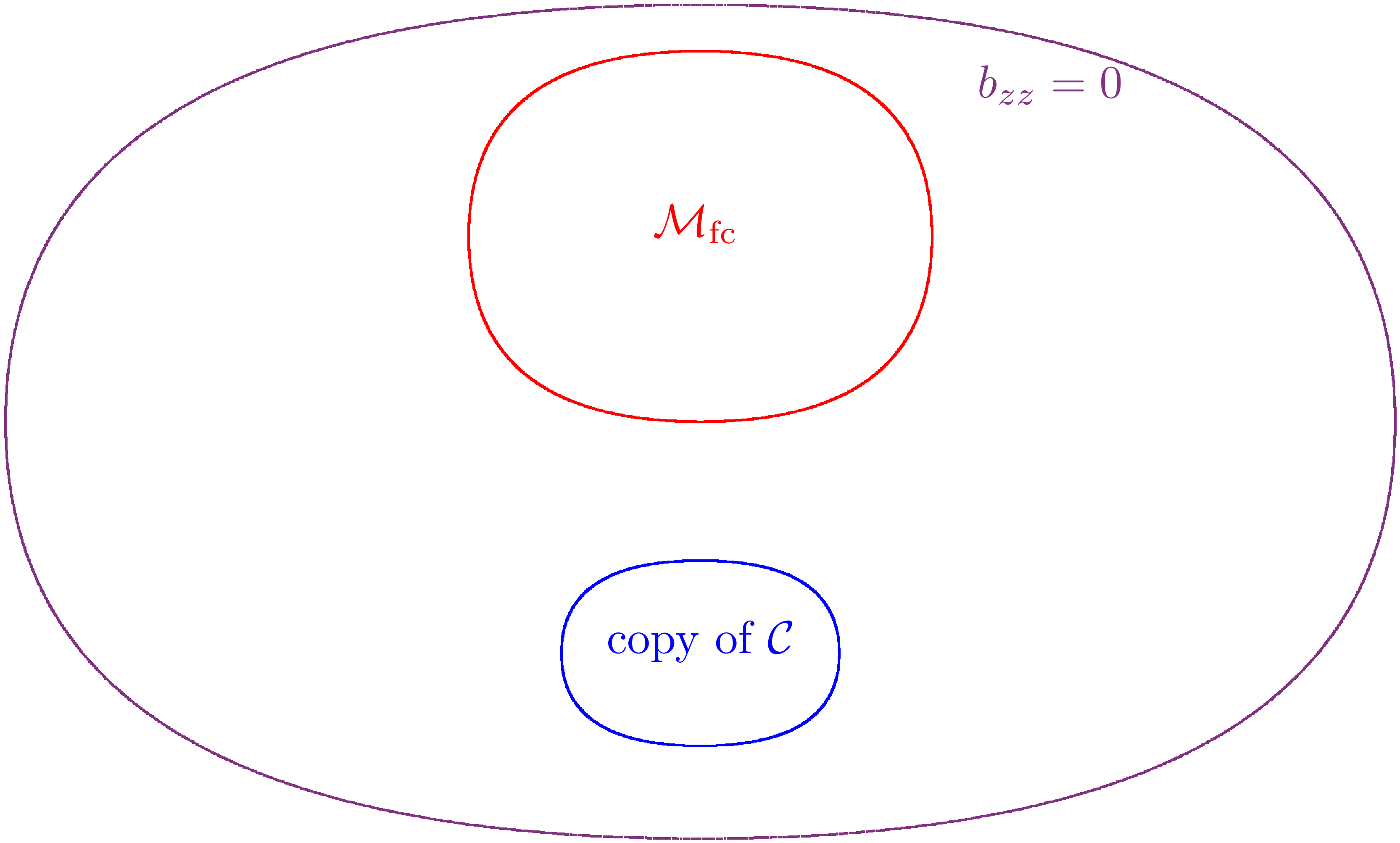
T = T-duality (mirror symmetry) twist
γ = some isometry twist



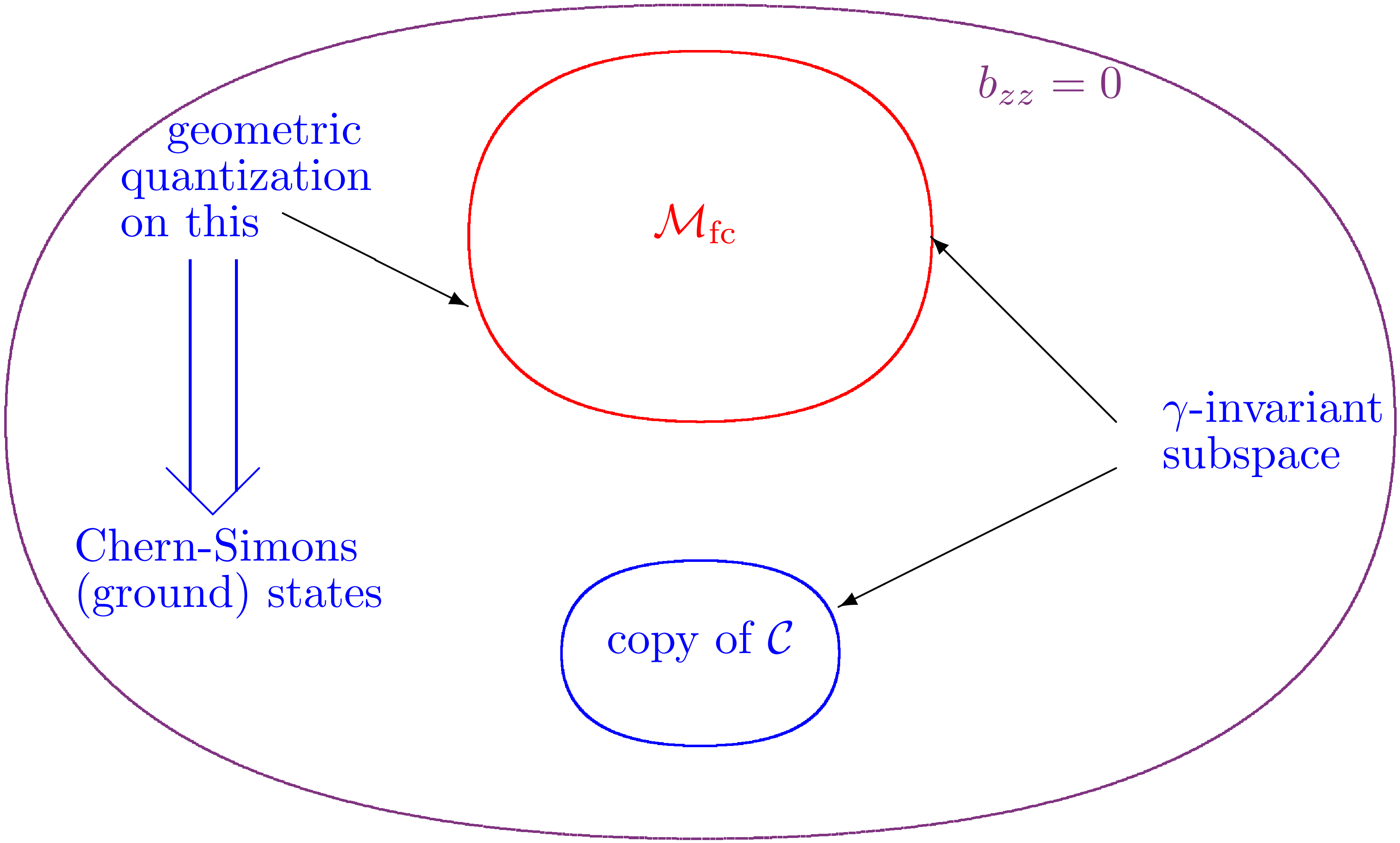
IR?

Geometric quantization on γ-invariant subspace???

The fiber over $b_{zz} = 0$ ......



The fiber over $b_{zz} = 0$ ......



Conclusions

- Compactification of $N = 4$ $U(n)$ SYM on S^1 with an S-duality twist, at a self-dual τ seems to give a topological 2+1D QFT in IR for n sufficiently small;
- Number of (ground) states on T^2 can be computed by string dualities;
- Number of (ground) states on $\mathcal{C}_h$ $(h > 1)$ could be computed if we could determine the signs in the action of S-duality on $H^*(\mathcal{M}_H)$;

Open questions

- What is this topological 2+1D theory?
- Wilson lines?
- Mirror symmetry twist and geometric quantization?
- Nonlocal topological structure from the kernel $\mathcal{S}(A, A_D)$? (Simple argument suggests that correlation functions of pairs of Wilson lines is proportional to the linking number.)
- Can we extract any new clues about S-duality from this?

Thank you!

Title