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21st Feb. '03 Ph 105 ANALYTICAL MECHANICS H.W. # 2

1. PROBLEM 1-6(a) Hand & Finch

Let $\mathcal{L}' = \alpha\mathcal{L} + \beta$ be a newly defined Lagrangian.
Here α, β are constants, $\mathcal{L}$ is the original Lagrangian.

Let $\{q_i\}$ be the generalized co-ordinates of the system.

Lagrange's EOM give us :

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \longrightarrow \star.$$

We need to prove :

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}'}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

Consider the left hand side,

$$\begin{aligned}
lhs &= \frac{d}{dt}\left(\frac{\partial \mathcal{L}'}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} \\
&= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_i}(\alpha\mathcal{L} + \beta)\right) - \frac{\partial}{\partial q_i}(\alpha\mathcal{L} + \beta) \\
&= \frac{d}{dt}\left(\alpha \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \alpha \frac{\partial \mathcal{L}}{\partial q_i} \\
&= \alpha\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i}\right] \\
&= 0 \quad \ldots \text{ from eqn. } \star. \\
&= rhs.
\end{aligned}$$

Since $\mathcal{L}'$ also satisfies the Lagrange's eqn. of motion for q_i, we say it is physically equivalent.

2. PROBLEM 1-6(b) Hand & Finch

Let $F(q_i, t)$ be a function only of the generalized co-ordinates $\{q_i\}$ and time. Consider a new Lagrangian $\mathcal{L}'$ given by

$\mathcal{L}' = \mathcal{L} + \frac{dF}{dt}$ where $\mathcal{L}$ is the original Lagrangian.

T.P.T. $\frac{d}{dt}\left(\frac{\partial \mathcal{L}'}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}'}{\partial q_i} = 0 \longrightarrow \star.$

<u>SOLN</u>: First consider $\frac{dF}{dt}$.

$$\frac{dF}{dt} = \sum_i \frac{\partial F}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial F}{\partial t} \quad \ldots \left[\text{chain rule where } F \text{ is a function of } \{q_i\}, t \text{ only}\right]$$

$$= \sum_i \dot{q}_i \frac{\partial F}{\partial q_i} + \frac{\partial F}{\partial t} \longrightarrow ①.$$

Consider lhs of $\star$.

$$\text{lhs} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}'}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}'}{\partial q_i}$$

$$= \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_i}\left(\mathcal{L} + \frac{dF}{dt}\right)\right] - \frac{\partial}{\partial q_i}\left(\mathcal{L} + \frac{dF}{dt}\right)$$

$$= \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \frac{\partial}{\partial \dot{q}_i}\left(\frac{dF}{dt}\right)\right] - \left(\frac{\partial \mathcal{L}}{\partial q_i} + \frac{\partial}{\partial q_i}\left(\frac{dF}{dt}\right)\right)$$

$$= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} + \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_i}\left(\frac{dF}{dt}\right)\right] - \frac{\partial}{\partial q_i}\left(\frac{dF}{dt}\right)$$

$$= 0 + \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_i}\left(\frac{dF}{dt}\right)\right] - \frac{\partial}{\partial q_i}\left(\frac{dF}{dt}\right) \quad \ldots \left[\because \mathcal{L} \text{ is original Lagrangian}\right]$$

$$= \frac{d}{dt}\left(\frac{\partial F}{\partial q_i}\right) - \frac{\partial}{\partial q_i}\left(\frac{dF}{dt}\right) \quad \ldots \left[\text{Using eqn. ① for } \frac{dF}{dt} \text{ and noting that } F \text{ isn't a function of } \dot{q}_i \text{ explicitly}\right]$$

$$= 0 \quad \ldots \left[\text{Commuting the derivatives of a cts. function } F\right]$$

Q.E.D. $\therefore$ $\mathcal{L}, \mathcal{L}'$ are physically equivalent.

3. PROBLEM 1-7 Hand & Finch

(a)(i) For a free particle space is empty and uniform. There is nothing that distinguishes one pt. in space over the other. Hence, due to this uniformity or homogeneity of space for a free particle, the Lagrangian is independant of the co-ordinates x, y, z.

(ii) For a free particle, space is also isotropic. No particular direction is more important than another, when space is empty. Thus the Lagrangian cannot depend on any particular velocity component like v_x, v_y or v_z.

b) I think the solution to this problem is fairly subtle and intricate, so I hope you will be patient with my modest attempts at explaining it.

Let me begin by stressing the existance of the 'tangent bundle' independant of the co-ordinate system used to describe it. The 'tangent bundle', as Prof. Hořava once mentioned in class, is the collection of all pts. $(\vec{r}, \vec{v})$ where $\vec{r}$ is the position and $\vec{v}$ is the velocity vector. So for our free particle, it would be the set of all positions with all velocity vectors at that position. The Lagrangian is a scalar function on this space, since it's a function of position (the q's) and velocity vectors (the $\dot{q}$'s). This space with it's Lagrangian exists independant of the co-ordinates we choose to describe the space.

Now the fellow sitting in frame K sets up a co-ordinate system which assigns numbers x, y, z and velocities v_x, v_y, v_z to every pt. in the above space. Using arguments (a)(i), (ii) he decides the Lagrangian at a pt. in the above space should be v^2 where v is the magnitude of the velocity vector at the chosen pt. in the tangent bundle in his

co-ordinate system. It's in the assignment of the Lagrangian to every pt. in the tangent bundle that the physics of the problem is sorted out. This is because, once a Lagrangian is assigned to the tangent bundle, the EOM fix the dynamical behavior of the system.

So we have $\mathcal{L}(p) = v^2$ where p is a pt. in the tangent bundle, whose co-ordinates in frame K are $(\vec{r}_K, \vec{v}_K)$ and $v^2 = |\vec{v}|^2$.

Now another fellow comes along with his own co-ordinate system K'. He is moving with a velocity $-\vec{v}_0$ w.r.t. the fellow in K. If he looks at the pt. p in his co-ordinate system, it will be $(\vec{r}_{K'}, \vec{v} + \vec{v}_0)$ where $\vec{r}_{K'}$ will be is some way related to $\vec{r}_K, \vec{v}_0$, time and that way is fairly easy to figure out but not of interest to us.

Since the Lagrangian is dependant on p and not on the co-ordinate, it can't change for K', it must just be expressed in K co-ordinates, which are $\cancel{\vec{r}_{K'}, \vec{v}_{K'}}$ $(\vec{r}_{K'}, \vec{v}_{K'})$ and have the $\therefore$ ~~$\mathcal{L}(p) = v^2$~~ relationship expressed above with the co-ordinates of K $\Rightarrow$

$$\begin{aligned}
\therefore \mathcal{L}(p) &= v^2 \\
&= |\vec{v}_{K'} - \vec{v}_0|^2 \quad \ldots \text{[expressing things in K' co-ordinates]} \\
&= |\vec{v}' - \vec{v}_0|^2 \quad \ldots \text{[In the problem we've called } \vec{v}_{K'} \text{ as } \vec{v}'\text{]} \\
&= (v')^2 - 2\vec{v}' \cdot \vec{v}_0 + v_0^2 \quad \ldots \text{[} v', v_0 \text{ are magnitudes of } \vec{v}', \vec{v}_0\text{]}
\end{aligned}$$

However, from Problem 1 and 2 we've learned that the Lagrangian at pt. p is not completely unique. We are free to add constants and total derivates of functions of the co-ordinates and time. So the fellow in K' decides to add the constant $-v_0^2$ and the total time derivative of the function $F = 2\vec{r}_{K'} \cdot \vec{v}_0$ i.e. $\frac{dF}{dt} = 2\frac{d}{dt}(\vec{r}_{K'} \cdot \vec{v}_0) = 2\vec{v}' \cdot \vec{v}_0$

In other words the Lagrangian at pt. p, ~~mig~~ expressed in K' co-ordinates might as well be

$$\mathcal{L}(p) = (v')^2 \quad !!!$$

That means to K', the Lagrangian at a pt. p looks just like the square of the magnitude of the velocity IN HIS FRAME. It has the same 'form' as the Lagrangian assigned to p by the fellow in K. But the Lagrangian contains all the physics, so if it has the same form, physics must look the same to them.

This shows the equivalence between two inertial frames, well, for a free particle at least!!!

4. PROBLEM 1-12 Hand & Finch

(i) Consider a free particle moving in a plane. Let it's polar co-ordinates and cartesian co-ordinates be given by (r, θ) & (x, y) respectively.

$\therefore$ We have the relationship,

$$x = r\cos\theta \Rightarrow \dot{x} = \dot{r}\cos\theta + r(-\sin\theta)\dot{\theta} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$$
$$y = r\sin\theta \Rightarrow \dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}$$

As talked about in problem 3., for a free particle

$\mathcal{L} = \frac{1}{2}mv^2$ ... [the $\frac{1}{2}m$ is just a proportionality constant]

$$= \tfrac{1}{2}m\,(\dot{x}^2 + \dot{y}^2)$$
$$= \tfrac{1}{2}m\left[(\dot{r})^2 + (r\dot{\theta})^2\right] \longrightarrow (\star)$$

(ii) Now suppose the particle does indeed have forces acting on it. We use the 'Golden Rule #1' from Hand & Finch, eqn. 1.54.

$$\mathcal{F}_k = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}$$

Here - $\mathcal{F}_k$ is the kth component of the generalized force and can be thought of as ma_k, the kth component of generalized acceleration.

- T is the kinetic energy of the particle and is simply the Lagrangian for a free particle, given by $\star$.
- q_k are the generalized co-ordinates which in our case will be r, θ

For the 'r' co-ordinate,

$$ma_r = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = m\ddot{r} - m\dot{\theta}^2 r = m(\ddot{r} - r\dot{\theta}^2) \longrightarrow (1)$$

For the θ co-ordinate,

$$m a_\theta = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) - \frac{\partial T}{\partial \theta} = \frac{d}{dt}(m r^2 \dot\theta) - 0 = m\, 2r\dot r\dot\theta + m r^2 \ddot\theta = m(r^2\ddot\theta + 2r\dot r\dot\theta)$$

$$\therefore m a_\theta = m(r^2\ddot\theta + 2r\dot r\dot\theta) \longrightarrow (2)$$

From (1), (2) we get

$$\vec a = a_r \vec e_r + a_\theta \vec e_\theta$$
$$= (\ddot r - r\dot\theta^2)\vec e_r + (r^2\ddot\theta + 2r\dot r\dot\theta)\vec e_\theta$$

Note that $\vec e_r$ and $\vec e_\theta$ are not necessarily unit vectors and, in general, can be dimensionful quantities. Just as F_k, the generalized k^{th} component of force, need not have dimensions of force, a_k needn't have dimensions of acceleration. Thus, the dimensions of $\vec e_k$ get determined s.t. $a_k \vec e_k$ in the above expression has the usual dimensions of acceleration, namely $[L]/[T]^2$. Thus $\vec e_\theta = 1/r\, \hat e_\theta$ since $\hat e_\theta$ is a dimensionless unit vector and r is the only length scale. Similarly, $\vec e_r = \hat e_r$.

$$\therefore \vec a = (\ddot r - r\dot\theta^2)\hat e_r + (r\ddot\theta + 2\dot r\dot\theta)\hat e_\theta .$$

5. PROBLEM 5

Consider a dynamical system with Lagrangian $\mathcal{L}$.

Let $\{q_i, \dot{q}_i, t\}$ and $\{\tilde{q}_i, \dot{\tilde{q}}_i, \tilde{t}\}$ be two sets of co-ordinates for this system, with the transformation,

$$\tilde{q}_j = \tilde{q}_j(q_i, t)$$

$$\dot{\tilde{q}}_j = \sum_i \frac{\partial \tilde{q}_j}{\partial q_i} \dot{q}_i + \frac{\partial \tilde{q}_j}{\partial t}$$

$$\tilde{t} = t.$$

Since the Lagrangian is a scalar function, the co-ordinate transformation leads to,

$$\mathcal{L}(q_i, \dot{q}_i, t) = \mathcal{L}\left(\tilde{q}_j(q_i, t),\ \dot{\tilde{q}}_j(q_i, \dot{q}_i, t),\ \tilde{t}(t)\right)$$

(i) $$\therefore \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \sum_j \left[\frac{\partial \mathcal{L}}{\partial \tilde{q}_j} \frac{\partial \tilde{q}_j}{\partial \dot{q}_i} + \frac{\partial \mathcal{L}}{\partial \dot{\tilde{q}}_j} \frac{\partial \dot{\tilde{q}}_j}{\partial \dot{q}_i}\right] + \frac{\partial \mathcal{L}}{\partial \tilde{t}} \frac{\partial \tilde{t}}{\partial \dot{q}_i}$$

However, the $\tilde{q}_j$ and $\tilde{t}$ are not explicit functions of $\dot{q}_i$

$$\therefore \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \sum_j \frac{\partial \mathcal{L}}{\partial \dot{\tilde{q}}_j} \frac{\partial \dot{\tilde{q}}_j}{\partial \dot{q}_i} = \sum_j \frac{\partial \mathcal{L}}{\partial \dot{\tilde{q}}_j} \frac{\partial \tilde{q}_j}{\partial q_i} \quad \cdots \text{[cancellation of dots]}$$

$$\therefore \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \sum_j \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\tilde{q}}_j}\right) \frac{\partial \tilde{q}_j}{\partial q_i} + \frac{\partial \mathcal{L}}{\partial \dot{\tilde{q}}_j} \frac{d}{dt}\left(\frac{\partial \tilde{q}_j}{\partial q_i}\right)\right] \longrightarrow (1)$$

(ii) Also $$\frac{\partial \mathcal{L}}{\partial q_i} = \sum_j \left[\frac{\partial \mathcal{L}}{\partial \tilde{q}_j} \frac{\partial \tilde{q}_j}{\partial q_i} + \frac{\partial \mathcal{L}}{\partial \dot{\tilde{q}}_j} \frac{\partial \dot{\tilde{q}}_j}{\partial q_i}\right] + \frac{\partial \mathcal{L}}{\partial \tilde{t}} \frac{\partial \tilde{t}}{\partial q_i}$$

Again $\tilde{t}$ isn't an explicit function of q_i gives,

$$\frac{\partial \mathcal{L}}{\partial q_i} = \sum_j \left[\frac{\partial \mathcal{L}}{\partial \tilde{q}_j} \frac{\partial \tilde{q}_j}{\partial q_i} + \frac{\partial \mathcal{L}}{\partial \dot{\tilde{q}}_j} \frac{\partial \dot{\tilde{q}}_j}{\partial q_i} \right]$$

$$= \sum_j \left[\frac{\partial \mathcal{L}}{\partial \tilde{q}_j} \frac{\partial \tilde{q}_j}{\partial q_i} + \frac{\partial \mathcal{L}}{\partial \dot{\tilde{q}}_j} \frac{d}{dt}\left(\frac{\partial \tilde{q}_j}{\partial q_i} \right) \right] \cdots \left[\begin{array}{l} \text{commuting total time} \\ \text{derivative with partial } q_i \end{array} \right]$$

Using this result in combination with ①,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{L}}{\partial q_i} = \sum_j \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\tilde{q}}_j} \right) \frac{\partial \tilde{q}_j}{\partial q_i} - \frac{\partial \mathcal{L}}{\partial \tilde{q}_j} \frac{\partial \tilde{q}_j}{\partial q_i}$$

$$= \sum_j \frac{\partial \tilde{q}_j}{\partial q_i} \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\tilde{q}}_j} \right) - \frac{\partial \mathcal{L}}{\partial \tilde{q}_j} \right]$$

Q.E.D.