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21st APRIL '03 Ph 105 H.W. # 8

1. PROBLEM 6.3 HF

Consider a 1-D problem with co-ordinates (z, p) in phase space and a potential $V(z)$.

$$\therefore \quad H(z,p) = \frac{p^2}{2m} + V(z)$$

Consider a new set of co-ordinates $(\mathcal{Z}, \mathcal{P})$ s.t.
$\mathcal{Z} = z - D(t)$ for some function D of time t.

Let such a transformation be generated by $F_2(z, \mathcal{P}, t)$

$\therefore$ The eqns. of transformation are :-

$$p = \frac{\partial F_2}{\partial z} \longrightarrow ① \quad ; \quad \mathcal{Z} = \frac{\partial F_2}{\partial \mathcal{P}} \longrightarrow ②$$

From ② we get $\dfrac{\partial F_2}{\partial \mathcal{P}} = \mathcal{Z} = z - D(t)$

$$\therefore \quad F_2(z, \mathcal{P}, t) = \mathcal{P}(z - D(t)) \longrightarrow ③$$

Substituting ③ in ①, $p = \mathcal{P}$

$\therefore$ Eqns. of transformation are : $\left.\begin{aligned} \mathcal{Z} &= z - D(t) \\ \mathcal{P} &= p \end{aligned}\right\} \longrightarrow \star$

$$\bar{H} = H + \frac{\partial F_2}{\partial t} = \frac{p^2}{2m} + V(z) + -\mathcal{P}\frac{dD}{dt}$$

$$= \frac{\mathcal{P}^2}{2m} + V(\mathcal{Z} + D(t)) - \mathcal{P}\dot{D} \longrightarrow ④$$

$\therefore$ For $(\mathcal{Z}, \mathcal{P})$, Hamilton's Equations give :-

$$\dot{\mathcal{P}} = -\frac{\partial \bar{H}}{\partial \mathcal{Z}} = -\frac{\partial V}{\partial \mathcal{Z}}$$

$$\dot{\mathcal{Z}} = -\frac{\partial \bar{H}}{\partial \mathcal{P}} = \frac{\mathcal{P}}{m} - \dot{D}$$

Suppose $\exists\ F_1(z, \mathcal{Z}, t) = -\mathcal{Z}\mathcal{P} + F_2(z, \mathcal{P}, t)$

$$= \mathcal{P}z - \mathcal{P}D - \mathcal{Z}\mathcal{P}$$

$$= \mathcal{P}(z - D) - \mathcal{Z}\mathcal{P}$$

where $\mathcal{Z} = \dfrac{\partial F_2}{\partial \mathcal{P}} = z - D$

Trying to eliminate $\mathcal{P}$ for the 'new' variable $\mathcal{Z}$ leads to $F_1(z, \mathcal{Z}, t) \equiv 0$!!

Thus a generating function of type F_1 is quite useless in this situation. One could've anticipated this, since F_1 gives the transformation eqn. $p = \dfrac{\partial F_1(z, \mathcal{Z}, t)}{\partial z}$. However, such a dependence is impossible for $\mathcal{Z} = z - D(t)$. The right hand side can never be a function of p.

2. PROBLEM 6.4 HF

Consider a freely falling body in 1-D with co-ordinates (z, p).

$$\therefore\ H = \frac{p^2}{2m} + mgz$$

Let F_4 be a time independent generating function to a new set of co-ordinates $(\mathcal{Z}, \mathcal{P})$ s.t. $\bar{H}(\mathcal{Z}, \mathcal{P}) = \mathcal{P}$.

$$\therefore \mathcal{P} = \bar{H} = H + \frac{\partial F_4}{\partial t} = H = \frac{p^2}{2m} + mgz$$

$$\therefore\ z = \frac{1}{mg}\left[\mathcal{P} - \frac{p^2}{2m}\right] = -\frac{\partial F_4}{\partial p} \quad \ldots \text{[by eqn. of transformation]}$$

$$\therefore\ F_4(p, \mathcal{P}) = \frac{1}{mg}\left[-\mathcal{P}p + \frac{p^3}{6m}\right]$$

$$\therefore \; Z = \frac{\partial F_4}{\partial P} = \frac{1}{mg}(-p)$$

$\therefore$ The explicit eqns. of transformation are :-

$$\left.\begin{aligned} Z &= \frac{-p}{mg} \\ P &= \frac{p^2}{2m} + mgz \end{aligned}\right\} \longrightarrow \star$$

In the new co-ordinates (Z, P), Hamilton's eqns. give :-

$$\dot{Z} = \frac{\partial \bar{H}}{\partial P} = \frac{\partial P}{\partial P} = 1$$

$\therefore \; Z = t + \text{constant}$ i.e. $Z \equiv$ time !

3. PROBLEM 6.10 HF

Consider a transformation from (q, p) to (Q, P) s.t.

$$Q = \log\left(\frac{\sin p}{q}\right) \quad , \quad P = q \cot p$$

a) $\therefore$

$$\begin{aligned} [Q, P] &= \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \\ &= \left[\frac{q}{\sin p}\left(\frac{-\sin p}{q^2}\right)\right]\left[q(-\operatorname{cosec}^2 p)\right] - \left[\frac{q}{\sin p}\frac{\cos p}{q}\right]\left[\cot p\right] \\ &= \operatorname{cosec}^2 p - \cot^2 p \\ &= 1 \end{aligned}$$

$\therefore$ The given transformation is cannonical.

b) T.S.T. $p\,dq - P\,dQ = d(pq + q\cot p)$

Here

$$\begin{aligned} \text{lhs} &= p\,dq - P\,dQ \\ &= p\,dq - (q\cot p)\left[\frac{\partial Q}{\partial q}dq + \frac{\partial Q}{\partial p}dp\right] \end{aligned}$$

$$= p\,dq - q\cot p\left[\frac{-1}{q}\,dq + \cot p\,dp\right]$$

$$= p\,dq + \cot p\,dq - q\cot^2 p\,dp \longrightarrow (1)$$

$$\begin{aligned} rhs &= d(pq + q\cot p) \\ &= q\,dp + p\,dq + d(q\cot p) \\ &= p\,dq + q\,dp + \cot p\,dq + q(-\operatorname{cosec}^2 p)\,dp \\ &= p\,dq + \cot p\,dq + q\,dp\left[1 - \operatorname{cosec}^2 p\right] \\ &= lhs \quad \ldots \left[\because\ 1 - \operatorname{cosec}^2 p = -\cot^2 p\right] \end{aligned}$$

Q.E.D.

c) Let $F_1(q, Q)$ be the generating function.

$$\therefore\ p = \frac{\partial F_1}{\partial q}, \quad P = -\frac{\partial F_1}{\partial Q}$$

$$\therefore\ \text{From (b)},\quad dF_1 = \frac{\partial F_1}{\partial q}\,dq + \frac{\partial F_1}{\partial Q}\,dQ$$

$$= p\,dq - P\,dQ$$

$$= d(pq + q\cot p)$$

$$\therefore\ F_1 = pq + q\cot p \quad \ldots \text{[Ignoring additive constant]}$$

$$= q\sin^{-1}(qe^{Q}) + q\frac{\sqrt{1 - q^2e^{2Q}}}{qe^{Q}} \quad \ldots \left[\text{Inverting } Q = \log\left(\frac{\sin p}{q}\right)\right]$$

$$= q\sin^{-1}(qe^{Q}) + e^{-Q}\sqrt{1 - q^2e^{2Q}}$$

4. PROBLEM 6.15 HF

a) Our co-ordinates are x, y, z p_x, p_y, p_z.

$$\therefore\ l_x = yp_z - zp_y, \quad l_y = zp_x - xp_z, \quad l_z = xp_y - yp_x.$$

$$\therefore\ [l_x, l_y] = \left[\frac{\partial l_x}{\partial x}\frac{\partial l_y}{\partial p_x} - \frac{\partial l_x}{\partial p_x}\frac{\partial l_y}{\partial x}\right] + \left[\frac{\partial l_x}{\partial y}\frac{\partial l_y}{\partial p_y} - \frac{\partial l_x}{\partial p_y}\frac{\partial l_y}{\partial y}\right] + \left[\frac{\partial l_x}{\partial z}\frac{\partial l_y}{\partial p_z} - \frac{\partial l_x}{\partial p_z}\frac{\partial l_x}{\partial z}\right]$$

$$\therefore [l_z, l_y] = \left[\frac{\partial l_z}{\partial z}\frac{\partial l_y}{\partial p_z} - \frac{\partial l_z}{\partial p_z}\frac{\partial l_y}{\partial z}\right] \quad \ldots \left[\because \frac{\partial l_\alpha}{\partial p_\alpha} = \frac{\partial l_\alpha}{\partial \alpha} = 0 \text{ for } \alpha = x,y,z\right]$$

$$= (-p_y)(-x) - (y)(p_x)$$

$$= x p_y - y p_x$$

$$= l_z .$$

The other two cyclical results follow similarly.

b) We evaluate only $[x, l_\beta]$, $[p_x, l_\beta]$, for $\beta = x, y, z$. The other results will follow by symmetry.

~~(i) Since l_x doesn't depend explicitly on x, p_x (i.e. $\frac{\partial l_x}{\partial x} = \frac{\partial l_x}{\partial p_x} = 0$),~~

~~$[x, l_x] = [p_x, l_x] = 0$~~

(i) Note that $\frac{\partial x}{\partial \alpha} = \begin{cases} 1 & \text{for } \alpha = x \\ 0 & \text{for } \alpha = y, z, p_x, p_y, p_z \end{cases}$ ... α is a co-ordinate

& $\frac{\partial p_x}{\partial \alpha} = \begin{cases} 1 & \text{for } \alpha = p_x \\ 0 & \text{for } \alpha = x, y, z, p_y, p_z \end{cases}$

$$\therefore [x, l_\beta] = \frac{\partial l_\beta}{\partial p_x}, \quad [p_x, l_\beta] = -\frac{\partial l_\beta}{\partial x} \quad \text{for } \beta = x, y, z$$

$\therefore$ Clearly the results follow,

$[x, l_x] = 0$, $[p_x, l_x] = 0$

$[x, l_y] = +z$, $[p_x, l_y] = +p_z$

$[x, l_z] = -y$, $[p_x, l_z] = -p_y$

5. PROBLEM 6.18

Consider a particle moving in a potential $V(q) = u\tan^2(aq)$.

(i) Let E be the total energy of a trajectory. $\therefore$ The turning pts. are given by the eqn.

$$E - V(q) = 0$$

$$\therefore \quad E = u\tan^2(aq)$$

$$\therefore \quad q_\pm = \pm\frac{1}{a}\tan^{-1}\left(\sqrt{\frac{E}{u}}\right)$$

(ii) From Eq. 6.94 HF,

$$I = \frac{1}{2\pi}\oint p\,dq \qquad \text{where} \quad p = \sqrt{2(E - V(q))}$$

$$= \frac{1}{2\pi}\, 2\int_{q_-}^{q_+}\sqrt{2}\sqrt{E - u\tan^2(aq)}\;dq$$

Let $\tan(aq) = x$. $\therefore$ $a\sec^2(aq)\,dq = dx$

$$\Rightarrow \quad dq = \frac{dx}{a(1+x^2)}$$

$$\therefore \quad I = \frac{\sqrt{2}}{\pi}\int_{-\sqrt{E/u}}^{+\sqrt{E/u}}\sqrt{E - ux^2}\;\frac{dx}{a(1+x^2)}$$

$$= \frac{\sqrt{2}}{\pi}\frac{\sqrt{u}}{a}\int_{-\alpha}^{+\alpha}\frac{\sqrt{\alpha^2 - x^2}}{1+x^2}\,dx \qquad \text{where} \quad \alpha = \sqrt{E/u}$$

Let $x = \alpha\sin\theta$ $\therefore$ $dx = \alpha\cos\theta\,d\theta$

$$\therefore \quad I = \frac{\sqrt{2}}{\pi}\frac{\sqrt{u}}{a}\int_{-\pi/2}^{+\pi/2}\frac{\sqrt{\alpha^2 - \alpha^2\sin^2\theta}}{1+\alpha^2\sin^2\theta}\;\alpha\cos\theta\,d\theta$$

$$= \frac{\sqrt{2}\cdot\sqrt{u}}{\pi a}\int_{-\pi/2}^{+\pi/2}\frac{\alpha^2\cos^2\theta\,d\theta}{1+\alpha^2\sin^2\theta}$$

Now the integrand can be written as

$$\frac{\alpha^2\cos^2\theta}{1+\alpha^2\sin^2\theta} = \frac{\alpha^2 - \alpha^2\sin^2\theta}{1+\alpha^2\sin^2\theta} = \frac{-1-\alpha^2\sin^2\theta}{1+\alpha^2\sin^2\theta} + \frac{(+1+\alpha^2)}{1+\alpha^2\sin^2\theta} = -1 + (1+\alpha^2)\frac{1}{1+\alpha^2\sin^2\theta}$$

$$\therefore \frac{\alpha^2\cos^2\theta}{1+\alpha^2\sin^2\theta} = -1 + (1+\alpha^2)\,\frac{\cancel{\sec^2\theta}}{\cancel{1+\alpha^2\tan^2\theta}}\;\frac{1}{\cos^2\theta+\sin^2\theta+\alpha^2\sin^2\theta}$$

$$= \frac{(1+\alpha^2)}{\cos^2\theta+(1+\alpha^2)\sin^2\theta} - 1$$

$$= \frac{(1+\alpha^2)\sec^2\theta}{1+(1+\alpha^2)\tan^2\theta} - 1$$

$$\therefore I = \frac{\sqrt{2}\sqrt{u}}{\pi a}\int_{-\pi/2}^{+\pi/2}\frac{\alpha^2\cos^2\theta}{1+\alpha^2\sin^2\theta}\,d\theta$$

$$= \frac{\sqrt{2}\cdot\sqrt{u}}{\pi a}\left[\int_{-\pi/2}^{+\pi/2}\left[\frac{(1+\alpha^2)\sec^2\theta}{1+(1+\alpha^2)\tan^2\theta}\right]d\theta - \int_{-\pi/2}^{+\pi/2}1\cdot d\theta\right]$$

$$= \frac{\sqrt{2}\cdot\sqrt{u}}{\pi a}\left[(1+\alpha^2)\int_{-\pi/2}^{+\pi/2}\frac{\sec^2\theta\,d\theta}{1+(1+\alpha^2)\tan^2\theta} - \theta\Big|_{-\pi/2}^{+\pi/2}\right]$$

$$= \frac{\sqrt{2}\cdot\sqrt{u}}{\pi a}\left[\frac{(1+\alpha^2)}{\sqrt{1+\alpha^2}}\int_{-\infty}^{+\infty}\frac{dy}{1+y^2} - \pi\right] \quad \ldots \text{ where } y=\sqrt{1+\alpha^2}\tan\theta,\ dy=\sqrt{1+\alpha^2}\sec^2\theta\,d\theta$$

$$= \frac{\sqrt{2}\cdot\sqrt{u}}{\pi a}\left[\sqrt{1+\alpha^2}\,\tan^{-1}y\Big|_{-\infty}^{+\infty} - \pi\right]$$

$$= \frac{\sqrt{2}\cdot\sqrt{u}}{\pi a}\left[\sqrt{1+\alpha^2}\;\pi - \pi\right]$$

$$= \frac{\sqrt{2}\cdot\sqrt{u}}{a}\left[\sqrt{1+\frac{E}{u}} - 1\right]$$

$$= \frac{\sqrt{2}}{a}\left(\sqrt{E+u} - \sqrt{u}\right)$$

$$\therefore \frac{aI}{\sqrt{2}} = \sqrt{E+u} - \sqrt{u}$$

Inverting eqn. to get $E(I)$, $E = \left(\frac{aI}{\sqrt{2}} + \sqrt{u}\right)^2 - u$

Since $H = E$, $H(I) = \left(\frac{aI}{\sqrt{2}} + \sqrt{u}\right)^2 - u$

$$\therefore \omega(I) = \frac{\partial H}{\partial I} = 2\left(\frac{aI}{\sqrt{2}} + \sqrt{u}\right)\cdot\frac{a}{\sqrt{2}}$$

$$\therefore \frac{\omega}{a\sqrt{2}} = \sqrt{E+u}$$