PHY 250 (P. Horava) Homework Assignment 1 Solutions
Grader: Uday Varadarajan

1. Problem 1.1 of Polchinski, Vol. 1:

   (a) Consider the usual relativistic point particle action in flat Minkowski space, use the parameterization $X^0 = \tau$ and then take the non-relativistic limit,

   $$S = -m \int d\tau \sqrt{-\dot{X}^\mu \dot{X}_\mu} = -m \int d\tau \sqrt{1 - \dot{X}^i \dot{X}_i} \approx \int d\tau \left[\frac{1}{2} m \dot{X}^i \dot{X}_i - m\right]. \tag{1.1}$$

   This clearly has the usual non-relativistic form with the potential energy being the rest mass.

   (b) Consider the NG action using the reparameterization invariance of the world sheet to fix $X^0 = \tau$ and $\partial_\sigma X^i \partial_\sigma X^i = 1$. In this gauge, $\partial_\sigma X^i = \mathbf{\hat{x}^i}$ is a unit vector tangent to the string, while $\partial_\tau X^i = \mathbf{v^i}$ is its velocity. Further, the integral over σ yields the length of the string. Thus, we can rewrite the NG action as

   $$\begin{aligned} S &= -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{-\det \partial_a X^\mu \partial_b X_\mu} \\ &= -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{(1 - \partial_\tau X^i \partial_\tau X_i)(\partial_\sigma X^i \partial_\sigma X_i) + (\partial_\sigma X^i \partial_\tau X_i)^2} \\ &= -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{1 - \mathbf{v} \cdot \mathbf{v} + (\mathbf{v} \cdot \mathbf{\hat{x}})^2}. \end{aligned} \tag{1.2}$$

   Now, the transverse velocity of the string is given by $\mathbf{v}_\perp = \mathbf{v} - (\mathbf{v} \cdot \mathbf{\hat{x}})\mathbf{\hat{x}}$, and since

   $$\mathbf{v}_\perp \cdot \mathbf{v}_\perp = \mathbf{v} \cdot \mathbf{v} - \mathbf{2}(\mathbf{v} \cdot \mathbf{\hat{x}})^{\mathbf{2}} + (\mathbf{v} \cdot \mathbf{\hat{x}})^{\mathbf{2}} = \mathbf{v} \cdot \mathbf{v} - (\mathbf{v} \cdot \mathbf{\hat{x}})^{\mathbf{2}}, \tag{1.3}$$

   we can rewrite the NG action as

   $$S = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{1 - \mathbf{v}_\perp \cdot \mathbf{v}_\perp}. \tag{1.4}$$

   Taking the non-relativistic limit, we find

   $$S = \int d\tau d\sigma \left[\frac{1}{4\pi\alpha'} \mathbf{v}_\perp \cdot \mathbf{v}_\perp - \frac{1}{2\pi\alpha'}\right]. \tag{1.5}$$

   Integrating over σ just yields a potential term proportional to the length of the string, while the coefficient of both terms yields a mass per unit length of $T = \frac{1}{2\pi\alpha'}$.