PHY 250 (P. Horava) Homework Assignment 3 Solutions
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1. Supersymmetrization of Problem 1.4 of Polchinski, Vol. 1:

   We consider the states of the open superstring spectrum at the first massive level in the GS formalism. In light cone gauge, the physical Hilbert space is constructed from two sets of oscillators, the S^α_{-m} and α^i_{-m} for $m > 0$, $i = 1 \ldots 8$, $\alpha = 1 \ldots 8$, acting on the vacuum. The vacuum state can be determined by the requirement that it must furnish a representation of the zero-mode algebra of the oscillators. This algebra is a tensor product of a Heisenberg algebra associated with the bosonic zero modes (x^μ and p^μ) and the Clifford-like algebra associated with the fermionic zero modes (S^α_0, obeying $\{S^\alpha_0, S^\beta_0\} = \delta^{\alpha\beta}$). Of course, the bosonic zero modes are taken care of by using on-shell momentum eigenstates $|k^\mu\rangle$ and we will ignore them from here on out. For the fermionic zero modes first note that the above Clifford algebra is not the usual Clifford algebra for Spin(8), as the S^α_0 transform as *spinors* rather than vectors. However, because of $SO(8)$ triality, this really doesn't matter, and this algebra is satisfied by 16×16 matrices of the same form as the usual Dirac matrices for spin(8), but with the indices rotated

$$S^\alpha_0 \sim \begin{pmatrix} 0 & \gamma^\alpha_{i\dot{\alpha}} \\ \gamma^\alpha_{\dot{\alpha}i} & 0 \end{pmatrix}. \tag{1.1}$$

   Note that the $\gamma^\alpha_{\dot{\alpha}i}$ are required to satisfy$\gamma^\alpha_{\dot{\alpha}i}\gamma^\beta_{i\dot{\beta}} + \gamma^\beta_{\dot{\alpha}i}\gamma^\alpha_{i\dot{\beta}} = 2\delta^{\alpha\beta}\delta_{\dot{\alpha}\dot{\beta}}$ and are explicitly constructed (with rotated indices) in Appendix 5.B in GSW. Examining the indices, we see that these matrices act on a direct sum of an $SO(8)$ vector $|j\rangle$ and the conjugate $SO(8)$ spinor $|\dot{\alpha}\rangle$. As discussed in class, these states are massless (the ordering ambiguities of the two sets of oscillators cancel) and correspond to a massless gauge boson and its gaugino superpartner. The first massive level with $\alpha' m^2 = 1$ is obtained by exciting the ground state with either α^i_{-1} or S^α_{-1}, resulting in the states,

$$\alpha^i_{-1}|j\rangle \text{ (64 bosons)} \quad S^\alpha_{-1}|j\rangle \text{ (64 fermions)} \tag{1.2}$$
$$S^\alpha_{-1}|\dot{\alpha}\rangle \text{ (64 bosons)} \quad \alpha^i_{-1}|\dot{\alpha}\rangle \text{ (64 fermions).} \tag{1.3}$$

   (a) First we consider the fermions, which arise from two $SO(8)$ vector-spinors of opposite chirality (Note: these are reducible as we will explain below, giving rise to a opposite chirality pairs of Majorana-Weyl spinors and gravitinos). Consider the tensor product of a Majorana $SO(9)$ spinor ψ_a (we can think of this as a direct sum of two Majorana-Weyl $SO(8)$ spinors, $\psi^a = \eta^\alpha \oplus \xi^{\dot{\alpha}}$) and an $SO(9)$ vector v_I, $I = i, 9$. This is a reducible representation of $SO(9)$, a **144**. We can extract a spinor "trace" by contracting the vector index with nine dimensional gamma matrices, $\chi_b \equiv \Gamma^I_{ab} v_I \psi^b$. Removing this trace, we are left with the irreducible gravitino, which is a $\mathbf{144} - \mathbf{16} = \mathbf{128}$. This is exactly the sum of the two fermionic $SO(8)$ reps above.

   The 128 bosons arise from an $SO(8)$ chiral bispinor and a two-tensor. A general two-tensor t^{ij} clearly just decomposes into a symmetric traceless two-tensor $g^{ij} = \frac{1}{2}(t^{ij} + t^{ji})$, an antisymmetric two-tensor $b^{ij} = \frac{1}{2}(t^{ij} - t^{ji})$, and a scalar t^{ii}. This is the decomposition $\mathbf{64} = \mathbf{35_v} + \mathbf{28} + \mathbf{1}$. Now we consider the $SO(8)$ irrep content of a chiral bispinor $\eta^\alpha \xi^{\dot{\alpha}}$. To do this, it is convenient to write η and ξ as 16 dimensional Dirac spinors,

$$\psi = \begin{pmatrix} \eta \\ 0 \end{pmatrix} \quad \chi = \begin{pmatrix} 0 \\ \xi \end{pmatrix}, \tag{1.4}$$

   satisfying the Weyl and Majorana conditions, $\frac{1}{2}(1 + \Gamma^9)\psi = \psi$, $\frac{1}{2}(1 - \Gamma^9)\chi = \chi$, and $\psi^\dagger = \psi^T$, $\eta^\dagger = \eta^T$. Note that with this choice of basis, $\Gamma^9 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$. To extract its irreps, we will consider "traces" of the above state via contractions with antisymmetrized products of 16 dimensional $SO(8)$ gamma matrices Γ^i (recall that the symmetric parts are trivial because of the Clifford algebra relation $\{\Gamma^i, \Gamma^j\} = 2\delta^{ij}$). In particular, we can construct the tensors,

$$A^{i_1 \ldots i_k} = \psi^\dagger \Gamma^{[i_1} \cdots \Gamma^{i_k]} \chi \tag{1.5}$$

Note that if k is even, using the Weyl conditions and the hermiticity of Γ^9,

$$\begin{aligned} A^{i_1 \dots i_k} &= \psi^\dagger \Gamma^{[i_1} \cdots \Gamma^{i_k]} \chi \\ &= \psi^\dagger \Gamma^{[i_1} \cdots \Gamma^{i_k]} \frac{1}{2}(1-\Gamma^9)\chi \\ &= \psi^\dagger \frac{1}{2}(1-\Gamma^9) \Gamma^{[i_1} \cdots \Gamma^{i_k]} \chi \\ &= (\frac{1}{2}(1-\Gamma^9)\psi)^\dagger \Gamma^{[i_1} \cdots \Gamma^{i_k]} \chi = 0. \end{aligned} \tag{1.6}$$

Thus, we must have k odd. Further, by inserting Γ^9 into the above expression, we obtain the duality relation, (up to a sign)

$$A^{i_1 \dots i_k} = \frac{\pm}{(8-k)!} \epsilon^{i_1 \dots i_8} A_{i_{k+1} \dots i_8}. \tag{1.7}$$

Thus, we only need to consider $k=1$ and $k=3$, so we get a vector and an antisymmetric three-tensor, corresponding to the decomposition, $\mathbf{64} = \mathbf{8_v} + \mathbf{56_v}$.

Now, consider an $SO(9)$ antisymmetric three-tensor A^{IJK} (an $\mathbf{84}$) and a symmetric, traceless two tensor, g^{IJ} (a $\mathbf{44}$). As $\mathbf{128} = \mathbf{84} + \mathbf{44}$, this pair has the right dimension to account for all the above states. To see that this is indeed the case, we can decompose them explicitly in terms of $SO(8)$ representations. The three-tensor decomposes into an $SO(8)$ antisymmetric three-tensor A^{ijk} (a $\mathbf{56_v}$), and an anti-symmetric two-tensor A^{ij9} (a $\mathbf{28}$). The symmetric, traceless two-tensor decomposes into an $SO(8)$ traceless two-tensor g^{ij} (a $\mathbf{35_v}$), a vector g^{i9} (an $\mathbf{8_v}$), and a scalar g^{99} (a $\mathbf{1}$). These match precisely with the $SO(8)$ representations we found above.

(b) Clearly, we have 128 bosons and 128 fermions so we have supersymmetry at this level.

(c) Since the massless modes of 11D supergravity must fit into irreps of $SO(9)$ with equal numbers of bosons and fermions, it is not hard to believe that these are precisely the same irreps that show up above. Indeed, the spectrum of 11D SUGRA consists of exactly a massless graviton, a three-form gauge field, and gravitino, just what we found above.

2. We will consider the first massless level of the open superstring using the NSR formalism in light cone gauge. The only subtlety here is making sure that we apply the GSO projection appropriately. Here, have two different sectors, the NS and R.

(a) The NS ground state is a tachyonic scalar $|0\rangle$ (again, we systematically ignore the momentum for notational simplicity) of mass $\alpha' m^2 = -1/2$. We generate the spectrum by acting upon it by using half-integrally moded fermionic oscillators b^i_{-r}, $r \in \mathbb{N} + 1/2$, and integrally moded bosonic oscillators α^i_{-n}, $n > 0$. Of course, for consistency, we must impose the GSO projection, which in the NS sector is a projection by the operator $G = (-1)^F = -(-1)^{\sum_r b^i_{-r} b^i_r}$. In particular, the ground state tachyon is odd and therefore projected out. The lowest lying state in this sector is a massless vector at level $N = 1/2$, $b^i_{-1/2} |0\rangle$ We now catalog all states at levels $N=1$ and $N = 3/2$ along with their eigenvalues under G.

$$\begin{aligned} \alpha^i_{-1} |0\rangle & \quad (N=1,\, \alpha' m^2 = 1/2,\, G=-1,\, \mathbf{8_v}) \\ b^i_{-1/2} b^j_{-1/2} |0\rangle & \quad (N=1,\, \alpha' m^2 = 1/2,\, G=-1,\, \mathbf{28}) \\ b^i_{-1/2} \alpha^j_{-1} |0\rangle & \quad (N=3/2,\, \alpha' m^2 = 1,\, G=+1,\, \mathbf{64} = \mathbf{1} + \mathbf{35_v} + \mathbf{28}) \\ b^i_{-1/2} b^j_{-1/2} b^k_{-1/2} |0\rangle & \quad (N=3/2,\, \alpha' m^2 = 1,\, G=+1,\, \mathbf{56_v}) \\ b^i_{-3/2} |0\rangle & \quad (N=3/2,\, \alpha' m^2 = 1,\, G=+1,\, \mathbf{8_v}) \end{aligned} \tag{1.8}$$

Eliminating the G odd states, we are left only with the states at $N = 3/2$, all of which are space-time bosons. We find an $SO(8)$ scalar ($\mathbf{1}$), vector ($\mathbf{8_v}$), antisymmetric two-tensor ($\mathbf{28}$), traceless symmetric two-tensor ($\mathbf{35}$), and an antisymmetric three-tensor $\mathbf{56_v}$, exactly as in the GS computation in Problem 1, $\mathbf{128} = \mathbf{1} + \mathbf{8_v} + \mathbf{28} + \mathbf{35_v} + \mathbf{56_v}$. Just as in that case, these fit into an $SO(9)$ traceless symmetric two-tensor and antisymmetric three-tensor.

The R ground state must admit a representation of the R zero modes, which form an $SO(8)$ Clifford algebra (up to normalization) $\{d_0^i, d_0^j\} = \delta^{ij}$. As is shown in Appendix 5.B of GSW, the smallest such representation is by 16×16 matrices (the same as those exhibited in Problem 1 above, but with rotated indices) operating on a 16 dimensional Majorana spinor. Note that this spinor is irreducible as a representation of the $SO(8)$ *Clifford algebra* but reducible as a representation of spin(8). Thus, the R ground state can be taken to be an $SO(8)$ Majorana spinor, which is the sum of two chiral Majorana-Weyl spinors $|a\rangle = |\alpha\rangle \oplus |\dot{\alpha}\rangle$. The R spectrum is generated by the integrally moded fermionic operators d_{-n}^i and bosonic operators α_{-n}^i, $n > 0$ acting on the R ground states. The GSO projection in the R sector acts as $\bar{\Gamma} = (-1)^F = \Gamma^9 (-1)^{\sum_{n>0} d_{-n}^i d_n^i}$, where $\Gamma^9 |\alpha\rangle = |\alpha\rangle$ and $\Gamma^9 |\dot{\alpha}\rangle = -|\dot{\alpha}\rangle$. Since the R ground states are massless, we see that the GSO projection will leave precisely a single massless Majorana-Weyl fermion, the superpartner to the photon, at the lowest level. At level one, before GSO projection, we have the states,

$$\begin{array}{ll} \alpha_{-1}^i |\alpha\rangle & (G = +1,\ \mathbf{64} = \mathbf{8_s} + \mathbf{56_s}) \\ d_{-1}^i |\alpha\rangle & (G = -1,\ \mathbf{64} = \mathbf{8_s} + \mathbf{56_s}) \\ \alpha_{-1}^i |\dot{\alpha}\rangle & (G = -1,\ \mathbf{64} = \mathbf{8_c} + \mathbf{56_c}) \\ d_{-1}^i |\dot{\alpha}\rangle & (G = +1,\ \mathbf{64} = \mathbf{8_c} + \mathbf{56_c}) \end{array} \tag{1.9}$$

Thus, we see that we are again left with two spinor-vectors of opposite chiralities, just as in the GS case, and they can be assembled into a spin 3/2 $SO(9)$ multiplet.

(b) We have found 128 bosons and 128 fermions just as before.

(c) Obviously, the same relation to the massless states of 11D supergravity obtains.

3. We consider the term,

$$\frac{i}{2\pi} \int_\Sigma d^2\sigma \ e \ \bar{\psi} \rho^a \omega_a^{AB} \rho^{AB} \psi. \tag{1.10}$$

Note that as $\rho^a = e_C^a \rho^C$ and using the Majorana condition, $\bar{\psi} = \psi^\dagger \rho^0 = \psi^T e_D^0 \rho^D$, we can rewrite this term as

$$\frac{i}{2\pi} \int_\Sigma d^2\sigma \ e \ e_C^a e_D^0 \omega_a^{AB} \ \psi^T \rho^D \rho^C \rho^{AB} \psi. \tag{1.11}$$

Since ψ is a two component real spinor, this term will vanish by fermi symmetry unless the matrix $\rho^D \rho^C \rho^{AB}$ is antisymmetric. However, in $1+1D$ we can find a basis in which both the Majorana and Weyl conditions can be simultaneously imposed, so $\rho^1 \rho^2 = \rho^3$ can be taken diagonal. Since $\rho^1 \rho^1 = \rho^2 \rho^2 = 1$ by the Clifford algebra, these facts imply that the product of four gamma matrices is always a symmetric matrix. Thus, this term vanishes.