PHY 250 (P. Horava) Homework Assignment 4 Solutions
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1. Problem 2.1 of Polchinski, Vol. 1:

   We first show that $\partial\bar\partial \log|z|^2 = 2\pi\delta^2(z,\bar z)$ using the analytic version of Stokes theorem. Since we know that $\partial\bar\partial\log|z|^2 = 0$ for $z \neq 0$,

$$\begin{aligned}\int d^2z\, f(z,\bar z)\partial\bar\partial\log|z|^2 &= f(0,0)\int d^2z\,\partial\bar\partial\log|z|^2 = if(0,0)\int dz\wedge d\bar z\,\partial\bar\partial\log|z|^2 \\ &= if(0,0)\oint d\bar z\,\bar\partial\log|z|^2 = -2\pi f(0,0)\oint\frac{d\bar z}{2\pi i}\frac{1}{\bar z} = 2\pi f(0,0).\end{aligned} \tag{4.1}$$

   Thus, we find that $\partial\bar\partial\log|z|^2 = 2\pi\delta^2(z,\bar z)$ as promised.

   Now, instead of using Stokes theorem, we can instead show the same relation by using a regulator starting with the expression,

$$\begin{aligned}\int d^2z\, f(z,\bar z)\partial\bar\partial\log|z|^2 &= f(0,0)\lim_{\epsilon\to0}\int d^2z\,\partial\bar\partial\log(|z|^2+\epsilon) = f(0,0)\lim_{\epsilon\to0}\int d^2z\,\partial\left(\frac{z}{|z|^2+\epsilon}\right) \\ &= f(0,0)\lim_{\epsilon\to0}\int d^2z\,\frac{\epsilon}{(|z|^2+\epsilon)^2} = f(0,0)\lim_{\epsilon\to0}\int_0^R 2rdr\int_0^{2\pi}d\theta\frac{\epsilon}{(r^2+\epsilon)^2} \\ &= f(0,0)\lim_{\epsilon\to0}\left(2\pi - \frac{2\pi\epsilon}{(R^2+\epsilon)}\right) = 2\pi f(0,0).\end{aligned} \tag{4.2}$$

   where we used $d^2z = 2rdrd\theta$, and we get the right result, $\partial\bar\partial\log|z|^2 = 2\pi\delta^2(z,\bar z)$.

2. Problem 2.3(a) of Polchinski, Vol. 1:

   We consider, without loss of generality, the first two vertex operators. From the OPE,

$$\left\langle :e^{ik_1\cdot X(z_1,\bar z_1)}::e^{ik_2\cdot X(z_2,\bar z_2)}:\cdots\right\rangle = |z_{12}|^{\alpha' k_1\cdot k_2}\left\langle :e^{i(k_1+k_2)\cdot X(z_2,\bar z_2)}(1+O(z_{12},\bar z_{12})):\cdots\right\rangle \tag{4.3}$$

   we would expect that the leading behavior of the amplitude as vertices 1 and 2 approach eachother, but all other vertices remain far apart, would be $|z_{12}|^{\alpha' k_1\cdot k_2}$ times the amplitude with $n-1$ vertex operators with the first one having momentum k_1+k_2. We'd like to verify this behavior using the explicit form of the amplitude,

$$\begin{aligned}\left\langle\prod_{i=1}^n :e^{ik_i\cdot X(z_i,\bar z_i)}:\right\rangle &= iC^X(2\pi)^D\delta^D(\sum_{i=1}^n k_i)\prod_{i<j}^n|z_{ij}|^{\alpha' k_i\cdot k_j}, \\ &= iC^X(2\pi)^D\delta^D(\sum_{i=1}^n k_i)\left(|z_{12}|^{\alpha' k_1\cdot k_2}\prod_{j=3}^n|z_{1j}|^{\alpha' k_1\cdot k_j}|z_{2j}|^{\alpha' k_2\cdot k_j}\right)\prod_{i<j=3}^n|z_{ij}|^{\alpha' k_i\cdot k_j},\end{aligned} \tag{4.4}$$

   where we've factored out the dependence on z_1 and z_2. Now, in the limit that we are taking, we note that

$$|z_{1j}| = |z_{12}+z_{2j}| = |z_{2j}|(1+O(z_{12},\bar z_{12})). \tag{4.5}$$

   Thus, we find that

$$|z_{12}|^{\alpha' k_1\cdot k_2}\prod_{j=3}^n|z_{1j}|^{\alpha' k_1\cdot k_j}|z_{2j}|^{\alpha' k_2\cdot k_j} = |z_{12}|^{\alpha' k_1\cdot k_2}\prod_{j=3}^n|z_{2j}|^{\alpha'(k_1+k_2)\cdot k_j}(1+O(z_{12},\bar z_{12})). \tag{4.6}$$

Plugging this back into the expression for the amplitude,

$$\left\langle \prod_{i=1}^{n} :e^{ik_i\cdot X(z_i,\bar{z}_i)}: \right\rangle = |z_{12}|^{\alpha' k_1\cdot k_2} iC^X (2\pi)^D \delta^D(\sum_{i=1}^{n} k_i) \prod_{l=3}^{n} |z_{2l}|^{\alpha'(k_1+k_2)\cdot k_l}(1+O(z_{12},\bar{z}_{12})) \prod_{i<j=3}^{n} |z_{ij}|^{\alpha' k_i\cdot k_j}$$
$$= |z_{12}|^{\alpha' k_1\cdot k_2} \left\langle :e^{i(k_1+k_2)\cdot X(z_2,\bar{z}_2)}(1+O(z_{12},\bar{z}_{12})): \prod_{i=3}^{n} :e^{ik_i\cdot X(z_i,\bar{z}_i)}: \right\rangle \tag{4.7}$$

which is just the behavior we expected.

3. Problem 2.10 of Polchinski, Vol. 1:

For this problem, we first note that the arguments we used to derive the XX OPE must be modified to account for the presence of the boundary. In particular, without a boundary, the Schwinger-Dyson equations,

$$0 = \int \mathcal{D}X \frac{\delta}{\delta X^\mu(z,\bar{z})} \left(X^\nu(z',\bar{z}')e^{-S}\right), \tag{4.8}$$

tell us that the XX expectation value is a Green's function, a solution to Poisson's equation,

$$\partial\bar{\partial}\left\langle X^\mu(z,\bar{z})X^\nu(z',\bar{z}')\right\rangle = -\pi\alpha'\eta^{\mu\nu}\delta^2(z-z'). \tag{4.9}$$

We can think of this as an electrostatics problem in 2D, where we are trying to find the electric potential at z due to a point charge at some position z'. We showed in Problem 1 (2.1 of Polchinski) that for the complex plane without a boundary, the solution is just $-\frac{\alpha'}{2}\eta^{\mu\nu}\log|z-z'|^2$. Now, suppose we introduce a boundary, which we take to be the real axis (so Im $z=0$). Far away from the boundary, we expect that the Schwinger-Dyson equations should be unchanged. Thus, we see that even in the presence of a boundary, the two point function must be a Green's function for z and z' not on the boundary. However, we must make sure that our solution respects the boundary conditions we impose. This additional requirement may be understood as arising from a modification to the Schwinger-Dyson equations due to restricting the range of integration to fields satisfying the boundary condition that we impose (for instance, by explicitly adding a delta functional fixing the b.c.'s). However, instead of pursuing this explictly via the path integral, we will use an analogy to electrostatics to solve this problem.

We consider Neumann b.c.'s, which means that we require that the normal derivative of X vanishes. Translating back to electrostatics, this is essentially requiring that the electric field normal to the boundary vanish. This is clearly not satisfied by the solution on the flat plane, as the electric field lines emanating radially from a single charge pierce through the boundary at Im $z=0$. However, it is easy to see that placing an image charge at the point $\bar{z}'$ implements our boundary condition perfectly. The boundary would then be midway between the two like charges, and therefore only have a tangential electric field. The electric potential due to the sum of these charges is given by,

$$\left\langle X^\mu(z,\bar{z})X^\nu(z',\bar{z}')\right\rangle_{D_2} = -\frac{\alpha'}{2}\eta^{\mu\nu}\log|z-z'|^2 - \frac{\alpha'}{2}\eta^{\mu\nu}\log|z-\bar{z}'|^2 \tag{4.10}$$

which obeys Poisson's equation everywhere in the upper half plane with Neumann boundary conditions. Note that since the image charge is always in the lower half plane, it never gives rise to any extra singularities for operators defined in the interior of the upper half plane. In particular, if z and z' are both in the interior of the upper half plane, the potential due to the image charge is harmonic (acting with the Laplacian simply gives us a delta-function which is never satisfied except for both z and z' on the boundary). Thus, we can see explicitly that the Schwinger-Dyson equations are unchanged in the interior. However, as an operator approaches the boundary, new divergences can appear due to collisions with the image charge.

In particular, as z approaches the boundary, the two charges coincide and give rise to double the original potential. Ordinary normal ordering just subtracts off the effect of the original charge,

$$:X^\mu(z,\bar{z})X^\nu(z',\bar{z}'):= X^\mu(z,\bar{z})X^\nu(z',\bar{z}') + \frac{\alpha'}{2}\eta^{\mu\nu}\log|z-z'|^2, \tag{4.11}$$

and so we find that for $z = y$ real,

$$\langle : X^\mu(y) X^\nu(z', \bar{z}') : \rangle_{D_2} = -\frac{\alpha'}{2} \eta^{\mu\nu} \log |y - \bar{z}'|^2 \tag{4.12}$$

is still singular as $z' \to y$! Thus, we need to modify our normal ordering scheme for operators defined on the boundary, to subtract off the contribution due to the image charge as well. This is done by defining boundary normal ordering by:

$${}^{\star}_{\star} X^\mu(y_1) X^\nu(y_2) {}^{\star}_{\star} = X^\mu(y_1) X^\nu(y_2) + \alpha' \eta^{\mu\nu} \log(y_1 - y_2)^2. \tag{4.13}$$

For arbitrary operators, we get

$${}^{\star}_{\star} \mathcal{F} {}^{\star}_{\star} = \exp\left(\int dy_1 dy_2 \frac{\alpha'}{2} \eta^{\mu\nu} \log(y_1 - y_2)^2 \frac{\delta}{\delta X^\mu(y_1)} \frac{\delta}{\delta X^\nu(y_2)} \right) \mathcal{F}. \tag{4.14}$$

Further, this yields the useful expression,

$${}^{\star}_{\star} \mathcal{F} {}^{\star\star}_{\star\star} \mathcal{G} {}^{\star}_{\star} = \exp\left(- \int dy_1 dy_2 \alpha' \eta^{\mu\nu} \log(y_1 - y_2)^2 \frac{\delta}{\delta X_F^\mu(y_1)} \frac{\delta}{\delta X_G^\nu(y_2)} \right) {}^{\star}_{\star} \mathcal{F}\mathcal{G} {}^{\star}_{\star}. \tag{4.15}$$

This easily yields the result that

$$\begin{aligned} {}^{\star}_{\star} e^{ik_1 \cdot X(y_1)} {}^{\star\star}_{\star\star} e^{ik_2 \cdot X(y_2)} {}^{\star}_{\star} &= \exp\left(-\alpha' \log(y_1 - y_2)^2 (ik_1) \cdot (ik_2) \right) {}^{\star}_{\star} e^{ik_1 \cdot X(y_1)} e^{ik_2 \cdot X(y_2)} {}^{\star}_{\star} \\ &= |y_1 - y_2|^{2\alpha' k_1 \cdot k_2} {}^{\star}_{\star} e^{ik_1 \cdot X(y_1)} e^{ik_2 \cdot X(y_2)} {}^{\star}_{\star} \end{aligned} \tag{4.16}$$

Now, we can translate between boundary normal ordering and the usual conformal normal ordering by using Eqn. 2.7.14 of Polchinski. First, we can extend the boundary normal ordering to an operator in the bulk just by subtracting off the extra term corresponding to the contribution of the image charge from our conformally normal ordered operator. Since this term is regular in the interior of the disk, this is a well defined normal ordering scheme. Then, the difference in the normal ordered products is just

$${}^{\star}_{\star} X^\mu(z_1, \bar{z}_1) X^\nu(z_2, \bar{z}_2) {}^{\star}_{\star} = : X^\mu(z_1, \bar{z}_1) X^\nu(z_2, \bar{z}_2) : + \frac{\alpha'}{2} \eta^{\mu\nu} \log |z_1 - \bar{z}_2|^2. \tag{4.17}$$

For a general operator, this difference (Eqn. 2.7.14) yields the relation,

$$: \mathcal{F} := \exp\left(- \int d^2 z_1 d^2 z_2 \frac{\alpha'}{4} \eta^{\mu\nu} \log |z_1 - \bar{z}_2|^2 \frac{\delta}{\delta X^\mu(z_1, \bar{z}_1)} \frac{\delta}{\delta X^\nu(z_2, \bar{z}_2)} \right) {}^{\star}_{\star} \mathcal{F} {}^{\star}_{\star}. \tag{4.18}$$

Since the boundary normal ordering eliminates the singularities appearing at the boundary, the only possible singular behavior of a conformal normal ordered operator will arise from the exponential of contractions in the above expression. In particular, considering the limit as Im $z \to 0$ of the following expression,

$$: e^{ik \cdot X(z, \bar{z})} := e^{\left(-\frac{\alpha'}{4} \log |z - \bar{z}|^2 (ik) \cdot (ik) \right)} {}^{\star}_{\star} e^{ik \cdot X(z, \bar{z})} {}^{\star}_{\star} = |2 \text{Im } z|^{\alpha' k^2 / 2} {}^{\star}_{\star} e^{ik \cdot X(z, \bar{z})} {}^{\star}_{\star} \tag{4.19}$$

we easily see that the conformal normal ordered vertex operator is non-singular on the boundary only if $k^2 > 0$, and generally diverges as $|2\text{Im} z|^{\alpha' k^2 / 2}$.

4. Problem 2.12 of Polchinski, Vol. 1:

The conserved charges α_m^μ (we consider only the holomorphic case - the antiholomorphic case is essentially identical) are just the Laurent coefficients of the holomorphic field $\partial X(z)$, (Eqn. 2.7.2a of Polchinski)

$$\alpha_m^\mu = \left(\frac{2}{\alpha'} \right)^{1/2} \oint \frac{dz}{2\pi} z^m \partial X^\mu(z). \tag{4.20}$$

Thus, as $Q = \oint \frac{dz}{2\pi i} j$, they are associated with the holomorphic currents,

$$j^\mu_m(z) = i \left(\frac{2}{\alpha'}\right)^{1/2} z^m \partial X^\mu(z). \tag{4.21}$$

Then, using the fact that (Eqn. 2.6.14 of Polchinski),

$$[Q_1, Q_2] = \oint \frac{dz_2}{2\pi i} \mathrm{Res}_{z_1 \to z_2} j_1(z_1) j_2(z_2), \tag{4.22}$$

and the OPE

$$\partial X^\mu(z_1) \partial X^\nu(z_2) \sim -\frac{\alpha'}{2z_{12}^2} \tag{4.23}$$

we find that

$$\begin{aligned}
[\alpha^\mu_m, \alpha^\nu_n] &= \oint \frac{dz_2}{2\pi i} \mathrm{Res}_{z_1 \to z_2} j^\mu_m(z_1) j^\nu_n(z_2) = -\left(\frac{2}{\alpha'}\right) \oint \frac{dz_2}{2\pi i} \mathrm{Res}_{z_1 \to z_2} z_1^m \partial X^\mu(z_1) z_2^n \partial X^\nu(z_2) \\
&= -\eta^{\mu\nu} \left(\frac{2}{\alpha'}\right) \oint \frac{dz_2}{2\pi i} \mathrm{Res}_{z_1 \to z_2} z_1^m z_2^n \left(-\frac{\alpha'}{2z_{12}^2}\right) \\
&= m\eta^{\mu\nu} \oint \frac{dz_2}{2\pi i} z_2^{n+m-1} \\
&= m\eta^{\mu\nu} \delta_{m,-n}.
\end{aligned} \tag{4.24}$$

The antiholomorphic part is essentially identical, and one can easily see that the antiholomorphic and holomorphic charges commute as the operator products $z^n \bar{z}^m \partial X \bar{\partial} X$ are non-singular. Now, we consider the commutation relation of the zero modes, p^μ and x^ν. To do this, we first notice that from above, we know that $p^\mu = \left(\frac{2}{\alpha'}\right)^{1/2} \alpha^\mu_0$ commutes with all the α^μ_m (and as its operator products with antiholomorphic currents are non-singular, with their antiholomorphic counterparts as well). Thus, using the fact that

$$X^\mu(z, \bar{z}) = x^\mu - i\frac{\alpha'}{2} p^\mu \log |z|^2 + i \left(\frac{\alpha'}{2}\right)^{1/2} \sum_{m \neq 0} \frac{1}{m} \left(\frac{\alpha^\mu_m}{z^m} + \frac{\tilde{\alpha}^\mu_m}{\bar{z}^m}\right), \tag{4.25}$$

we see that

$$[p^\mu, X^\nu(z, \bar{z})] = [p^\mu, x^\mu] \tag{4.26}$$

Then, we can use Eqn. 2.6.15 of Polchinski,

$$[Q, \mathcal{A}(z_2, \bar{z}_2)] = \mathrm{Res}_{z_1 \to z_2} j(z_1) \mathcal{A}(z_2, \bar{z}_2) \tag{4.27}$$

to compute this commutator. The current associated with p^μ is just $j_{p^\mu}(z) = \left(\frac{2}{\alpha'}\right)^{1/2} j^\mu_0(z) = i\left(\frac{2}{\alpha'}\right) \partial X^\nu(z)$, and using the OPE,

$$\partial X^\nu(z_1) X^\mu(z_2, \bar{z}_2) = -\eta^{\mu\nu} \frac{\alpha'}{2z_{12}} \tag{4.28}$$

we find that,

$$\begin{aligned}
[p^\mu, x^\nu] &= [p^\mu, X^\nu(z_2, \bar{z_2})] = i\mathrm{Res}_{z_1 \to z_2} \left(\frac{2}{\alpha'}\right) \partial X^\mu(z_1) X^\nu(z_2, \bar{z}_2) = -i\eta^{\mu\nu} \mathrm{Res}_{z_1 \to z_2} \frac{1}{z_{12}} \\
&= -i\eta^{\mu\nu}.
\end{aligned} \tag{4.29}$$

One can proceed similarly for the bc theory, the currents are easily read off

$$b_m = \oint \frac{dz}{2\pi i} z^{m+\lambda-1} b(z) \qquad c_n = \oint \frac{dz}{2\pi i} z^{n-\lambda} c(z), \tag{4.30}$$

and we can use the OPEs $b(z)c(0) \sim \frac{1}{z}$ to easily read off the anticommutation relations,

$$\begin{aligned}\{b_m, c_n\} &= \oint \frac{dz_2}{2\pi i}\mathrm{Res}_{z_1\to z_2} z_1^{m+\lambda-1} b(z_1) z_2^{n-\lambda} c(z_2) \\ &= \oint \frac{dz_2}{2\pi i}\mathrm{Res}_{z_1\to z_2} z_1^{m+\lambda-1} z_2^{n-\lambda}\left(\frac{1}{z_{12}}\right) = \oint \frac{dz_2}{2\pi i} z_2^{m+n-1} \\ &= \delta_{m,-n}.\end{aligned} \tag{4.31}$$

5. Problem 3.2 of Polchinski, Vol. 1:

(a) To find the number of components of a totally symmetric n-tensor in d-dimensions $T_{a_1\cdots a_n}$ define, we use the fact that the independent components of such a tensor are just the number of different sequences,

$$a_1 \le a_2 \le \cdots \le a_n \qquad a_i \in 1, \ldots, d. \tag{4.32}$$

Of course, for $d = 2$, this is trivial, as each such sequence is uniquely characterized by the number of 1's it has, and so there are precisely $n+1$ such sequences. Thus, the totally symmetric tensor has $n+1$ independent components. The condition that it be traceless requires that all contractions with a metric vanish,

$$g^{a_i a_j} T_{a_1\cdots a_n} = 0 \tag{4.33}$$

Now, by symmetry, this is equivalent to just the condition that

$$g^{a_1 a_2} T_{a_1 a_2\cdots a_n} = 0 \tag{4.34}$$

Further, by symmetry, these conditions are independent only for distinct sets of increasing $a_3, \ldots, a_n$. Thus, the relations form a symmetric $n-2$-tensor with $n-1$ independent components. This means that the symmetric traceless n-tensor has $n+1-(n-1) = 2$ independent components.

(b) We can define a differential operator which takes a traceless symmetric n-tensor into a traceless symmetric $n+1$ tensor by taking the symmetrized covariant derivative of the tensor, and subtracting off the trace. So for $T_{i_1\cdots i_n}$, define

$$(P_n T)_{a_0\cdots a_n} \equiv \frac{1}{2n}\sum_{i=1}^{n}\left(\nabla_{a_0} T_{a_1\cdots a_n} + \nabla_{a_i} T_{a_0 a_1\cdots \hat{a}_i\cdots a_n} - g_{a_0 a_i}\nabla_a T^a_{a_1\cdots\hat{a}_i\cdots a_n}\right). \tag{4.35}$$

This definition is clearly symmetric, and to see that it is traceless, we only need to consider contractions with $g^{a_0 a_j}$ for some j (all other traces vanish trivially since T is traceless and the metric is covariantly constant). Consider this trace term by term in the above sum. If $j = i$, then the first two terms are identical and cancel with the second since $g^{ab}g_{ab} = 2$ in two dimensions. If $j \ne i$, using the fact that the metric is covariantly constant, we note that

$$g^{a_0 a_j}\nabla_{a_0} T_{a_1\cdots a_n} = \nabla_{a_0} T^{a_0}_{a_1\cdots\hat{a}_j\cdots a_n} \tag{4.36}$$

$$g^{a_0 a_j}\nabla_{a_i} T_{a_0 a_1\cdots\hat{a}_i\cdots a_n} = \nabla_{a_i} g^{a_0 a_j} T_{a_0 a_1\cdots\hat{a}_i\cdots a_n} = 0 \tag{4.37}$$

$$g^{a_0 a_j} g_{a_0 a_i}\nabla_a T^a_{a_1\cdots\hat{a}_i\cdots a_n} = \nabla_a T^a_{a_1\cdots\hat{a}_j\cdots a_n} \tag{4.38}$$

and it is easy to see that the first and third terms are identical and therefore cancel, showing that the trace does vanish.

(c) We define the P_n^T as

$$(P_n^T T)_{a_1\cdots a_{n-1}} \equiv -\nabla_a T^a_{a_1\cdots a_{n-1}} \tag{4.39}$$

Again, as g is covariantly constant, this is traceless.

(d) Now, consider the n-tensor $T_{a_1\cdots a_n}$ and the $n+1$-tensor $S_{a_0\cdots a_n}$. Introduce the inner product,

$$(T, T') = \int d^2\sigma g^{1/2} T^{a_1\cdots a_n} T'_{a_1\cdots a_n}. \tag{4.40}$$

Then, we find that,

$$\begin{aligned}(S, P_n T) &= \frac{1}{2n}\int d^2\sigma g^{1/2} S^{a_0\cdots a_n} \sum_{i=1}^{n} \left(\nabla_{a_0} T_{a_1\cdots a_n} + \nabla_{a_i} T_{a_0 a_1\cdots \hat{a}_i\cdots a_n} - g_{a_0 a_i}\nabla_a T^{a}_{a_1\cdots\hat{a}_i\cdots a_n}\right)\\ &= \frac{1}{2n}\int d^2\sigma g^{1/2} S^{a_0\cdots a_n}(2n)\nabla_{a_0} T_{a_1\cdots a_n}\\ &= -\int d^2\sigma g^{1/2}\nabla_{a_0} S^{a_0\cdots a_n} T_{a_1\cdots a_n}\\ &= (P_n^T S, T).\end{aligned} \tag{4.41}$$

In going from the first to the second line, the tracelessness of S kills the third expression in each term, and its symmetry equates all the remaining $2n$ terms. In the third line, we've used integration by parts - the ordinary derivative acting on $g^{1/2}$ exactly produces the extra term needed to get the full covariant derivative acting on S.

6. Problem 3.8 of Polchinski, Vol. 1:

We would like to understand the conditions under which the renormalized vertex operators for closed string massless states of the bosonic string in the Polyakov formalism are Weyl invariant on a curved world sheet.

This will tell us which vertex operators we may consistently include in calculating scattering amplitudes, and thereby characterize the possible massless asymptotic physical states in this model. In addition we can imagine exponentiating these vertex operators to obtain coherent states which can be inserted into the path integral to alter the background in which the strings propagate. If the alteration to the background is small (so the wavelength or characteristic length scale R_c of the variations in the fields represented by the operators is large compared to $\sqrt{\alpha'}$), we may expand the exponential in powers of $\sqrt{\alpha'}/R_c$. The Weyl invariance of the vertex operators shows us that the exponential insertion itself is Weyl invariant to first order. Thus, this calculation simultaneously gives us knowledge to first order about consistent variations of the *background* in which the strings propagate. Of course, these are just the equations of motion of the background fields to lowest order.

In order to do this, we generalize the conformal normal ordering prescription to curved backgrounds by using a manifestly Diff invariant renormalization prescription, where the renormalized operator is given by (Eqn. 3.6.5 of Polchinski),

$$[\mathcal{F}]_r = \exp\left(\frac{1}{2}\int d^2\sigma_1 d^2\sigma_2 \Delta(\sigma_1,\sigma_2)\frac{\delta}{\delta X^\mu(\sigma_1)}\frac{\delta}{\delta X_\mu(\sigma_2)}\right)\mathcal{F}, \tag{4.42}$$

and we've defined,

$$\Delta(\sigma_1,\sigma_2) = \frac{\alpha'}{2}\log d^2(\sigma_1,\sigma_2), \tag{4.43}$$

where $d(\sigma_1,\sigma_2)$ is the geodesic distance between the points σ_1 and σ_2 on the worldsheet. Note that the geodesic distance is well defined only locally on compact spaces where many geodesics may connect a pair of points.[1]. However, as we are interested primarily in the short distance behavior

[1]Consider the north and south pole for S^2 embedded in $\mathbb{R}^3$ - all longitudes are geodesics. Moreover, for compact spaces with posive curvature metric, there exist geodesics which are not even local minima. For the sphere, take two points near eachother and consider the geodesic that corresponds to going around the wrong way - this is clearly not even locally a minimum of the distance function

of the operators, this local definition suffices and we can restrict ourselves to a coordinate patch small enough to pick a unique continuous definition of geodesic distance.

We note that while this renormalization prescription keeps Diff manifest, its explicit dependence on the metric through the geodesic distance induces non-trivial quantum corrections to the Weyl dependence of renormalized operators. Explicitly, the renormalization prescription gives rise to additional Weyl dependence of operators through the relation (Polchinski, 3.6.7)

$$\delta_W[\mathcal{F}]_r = [\delta_W \mathcal{F}]_r + \frac{1}{2}\int d^2\sigma_1 d^2\sigma_2 \delta_W \Delta(\sigma_1,\sigma_2)\frac{\delta}{\delta X^\mu(\sigma_1)}\frac{\delta}{\delta X_\mu(\sigma_2)}[\mathcal{F}]_r. \tag{4.44}$$

On a Riemann surface, we can always find local coordinates (via an appropriate Diff transformation) such that its metric is in conformal gauge. In conformal gauge, one can analyze the Weyl variation of the geodesic distance, and obtain the result (Polchinski, 3.6.15),

$$\partial_a \delta_W \Delta(\sigma_1,\sigma_2)|_{\sigma_1=\sigma_2=\sigma} = \frac{1}{2}\partial_a \delta\omega(\sigma) \tag{4.45}$$

$$\partial_a \partial_b' \delta_W \Delta(\sigma_1,\sigma_2)|_{\sigma_1=\sigma_2=\sigma} = \frac{1+\gamma}{2}\alpha' \nabla_a \partial_b \delta\omega(\sigma) \tag{4.46}$$

$$\nabla_a \partial_b \delta_W \Delta(\sigma_1,\sigma_2)|_{\sigma_1=\sigma_2=\sigma} = -\frac{\gamma}{2}\alpha' \nabla_a \partial_b \delta\omega(\sigma). \tag{4.47}$$

with $\gamma = -\frac{2}{3}$. Further, in conformal gauge, we use the fact that the Weyl variation of the Riemann tensor is given by (Polchinski, 3.3.5):

$$\delta_W(g^{1/2}R) = -2g^{1/2}\nabla^2\delta\omega \tag{4.48}$$

(to be posted in detail later)

7. Problem 3.9 of Polchinski, Vol. 1: (to be posted in detail later)