PHY 250 (P. Horava) Homework Assignment 5 Solutions
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1. Exercise 4.3 of Polchinski: (we use this method since it is significantly easier!)

   The Jacobi identity for the charges $L_m = L^{\mathrm{m}}_m + L^{\mathrm{g}}_m$, Q_B, and b_n, reads (this is obvious since these brackets are algebraic (anti-)commutators)

$$\{[Q_B, L_m], b_n\} - \{[L_m, b_n], Q_B\} - [\{b_n, Q_B\}, L_m] = 0. \tag{5.1}$$

   From Polchinski, 4.3.6,

$$\{Q_B, b_n\} = L_m = L^{\mathrm{m}}_m + L^{\mathrm{g}}_m, \tag{5.2}$$

   and using the fact that b_n is a Laurent mode of a Virasoro primary field $b(z)$ with weight $(2, 0)$ and therefore obeys (Polchinski, 2.6.24),

$$[L_m, b_n] = ((2-1)m - n)b_{m+n} = (m - n)b_{m+n}, \tag{5.3}$$

   we see that

$$\{[Q_B, L_m], b_n\} = \{(m - n)b_{m+n}, Q_B\} - [L_m, L_n] = (m - n)L_{m+n} - [L_m, L_n]. \tag{5.4}$$

   Thus, since $c^{\mathrm{g}} = -26$, we see that $\{[Q_B, L_m], b_n\} = 0$ iff the $c^{\mathrm{m}} = 26$. Note that since $[Q_B, L_m]$ has ghost number $N^{\mathrm{g}} = 1$, if it is non-zero, it must contain operators proportional to some c_n. Since $\{b_n, c_m\} = \delta_{n,-m}$, we can think of $b_n \sim \frac{\partial}{\partial c_{-n}}$ when acting via commutation or anticommutation. Thus, our result shows that $[Q_B, L_m]$ is independent of c_n for all n, and therefore vanishes when $c^{\mathrm{m}} = 26$.

   Now, the Jacobi identity for Q_B, Q_B, and b_n reads,

$$[\{Q_B, Q_B\}, b_n] = -[\{b_n, Q_B\}, Q_B] - [\{Q_B, b_n\}, Q_B] = 2[Q_B, L_m]. \tag{5.5}$$

   Thus, we see that $[\{Q_B, Q_B\}, b_n] = 0$ when $c^{\mathrm{m}} = 26$, and for the same reasons as above, as we expect $\{Q_B, Q_B\}$ to contain ghost modes c_n, this means $\{Q_B, Q_B\} = 0$ as well in the critical dimension.

2. Problem 4.5(a) of Polchinski, Vol. 1:

   For the closed string, BRST quantization instructs us to consider the Fock space of states satisfying the constraints (the first pair implies the second),

$$b_0|\psi\rangle = \tilde{b}_0|\psi\rangle = 0, \tag{5.6}$$
$$L_0|\psi\rangle = \tilde{L}_0|\psi\rangle = 0. \tag{5.7}$$

   For the closed string, we have,

$$L_0 = \frac{\alpha'}{4}(p^2 + m^2), \qquad \tilde{L}_0 = \frac{\alpha'}{4}(p^2 + \tilde{m}^2), \tag{5.8}$$

   where,

$$\frac{\alpha'}{4}m^2 = \sum_{n=1}^{\infty} n\left(N_{bn} + N_{cn} + \sum_{\mu=0}^{25} N_{\mu n}\right) - 1, \tag{5.9}$$
$$\frac{\alpha'}{4}\tilde{m}^2 = \sum_{n=1}^{\infty} n\left(\tilde{N}_{bn} + \tilde{N}_{cn} + \sum_{\mu=0}^{25} \tilde{N}_{\mu n}\right) - 1. \tag{5.10}$$

Now, we consider the reduced inner product $\langle \| \rangle$ which ignores the X^0 and ghost zero modes and consider states with $k^0 > 0$ determined by $\mathbf{k}$ and with appropriate ghost zero mode insertions. At lowest level, we have $N = \tilde{N} = 0$, and we have the state

$$|\psi_0\rangle = |0, 0; \mathbf{k}\rangle, \qquad -k^2 = -\frac{4}{\alpha'} = m^2 = \tilde{m}^2. \tag{5.11}$$

This state is clearly BRST invariant since all the lowering operators contain either a lowering operator or L_0 or $\tilde{L}_0$. Further, this implies that there are no exact states at this level, so the cohomology just consists of these states, the closed string tachyon.

Now consider all possible states at level $N = \tilde{N} = 1$. By the above formula, these are massless states of the form,

$$\begin{aligned} |\psi_1\rangle = \Big(&\alpha_{-1} \cdot e \cdot \tilde{\alpha}_{-1} + \beta \cdot \alpha_{-1}\tilde{b}_{-1} + \tilde{\beta} \cdot \tilde{\alpha}_{-1}b_{-1} + \gamma \cdot \alpha_{-1}\tilde{c}_{-1} + \tilde{\gamma} \cdot \tilde{\alpha}_{-1}c_{-1} \\ &+ B_{\tilde{c}}b_{-1}\tilde{c}_{-1} + \tilde{B}_c c_{-1}\tilde{b}_{-1} + Bb_{-1}\tilde{b}_{-1} + Cc_{-1}\tilde{c}_{-1}\Big) |0, 0; \mathbf{k}\rangle, \end{aligned} \tag{5.12}$$

with $-k^2 = 0$. Thus, we see that there are $26 \times 26 + 4 \times 26 + 4$ states at level 1 of the closed string. Taking their norms by using the hermiticity conditions of Polchinski 4.3.17, we see

$$\langle \psi_1 \| \psi_1\rangle = (e^* \cdot e + \beta^* \cdot \gamma + \gamma^* \cdot \beta + \tilde{\beta}^* \cdot \tilde{\gamma} + \tilde{\gamma}^* \cdot \tilde{\beta} + B^*_{\tilde{c}}\tilde{B}_c + \tilde{B}^*_c B_{\tilde{c}} + B^*C + C^*B)\langle 0, 0; \mathbf{k} \| 0, 0; \mathbf{k}\rangle \tag{5.13}$$

The states with positive norm consist of all spatially polarized e_{ij}, $i, j = 1, \ldots 25$, e_{00}, half the vector-like states corresponding to the appropriate linear combinations of the $\beta_\mu, \gamma_\mu, \tilde{\beta}_\mu, \tilde{\gamma}_\mu$, as well as half the scalar ghost states. This gives $25 \times 25 + 1 + 2 \times 26 + 2$ positive norm modes, and the remaining $2 \times 25 + 2 \times 26 + 2$ (timelike polarized e_{0i} and e_{i0} as well as the remaining vector and scalar ghost states) all have negative norm. The condition that these modes be BRST closed under both the holomorphic and antiholomorphic BRST operators gives rise to the conditions that (note that $k^2 = 0$ eliminates many of the c_0 contributions and the rest cancel),

$$\begin{aligned} 0 = Q_B|\psi_1\rangle &= \left(\sqrt{\frac{\alpha'}{2}}\,[c_{-1}k \cdot \alpha_1 + c_1 k \cdot \alpha_{-1}] + \alpha_{-1} \cdot \alpha_1 c_0 - b_{-1}c_0c_1 - c_{-1}c_0b_1 - c_0\right)|\psi_1\rangle \\ &= \sqrt{\frac{\alpha'}{2}}\Big(k \cdot e \cdot \tilde{\alpha}_{-1}c_{-1} + (k \cdot \beta)c_{-1}\tilde{b}_{-1} + (k \cdot \gamma)c_{-1}\tilde{c}_{-1} + \tilde{\beta} \cdot \tilde{\alpha}_{-1}k \cdot \alpha_{-1} \\ &\quad + B_{\tilde{c}}k \cdot \alpha_{-1}\tilde{c}_{-1} + B\tilde{b}_{-1}k \cdot \alpha_{-1}\Big)|0, 0; \mathbf{k}\rangle, \end{aligned} \tag{5.14}$$

$$\begin{aligned} 0 = \tilde{Q}_B|\psi_1\rangle &= \left(\sqrt{\frac{\alpha'}{2}}\,[\tilde{c}_{-1}k \cdot \tilde{\alpha}_1 + \tilde{c}_1 k \cdot \tilde{\alpha}_{-1}] + \tilde{\alpha}_{-1} \cdot \tilde{\alpha}_1\tilde{c}_0 - \tilde{b}_{-1}\tilde{c}_0\tilde{c}_1 - \tilde{c}_{-1}\tilde{c}_0\tilde{b}_1 - \tilde{c}_0\right)|\psi_1\rangle \\ &= \sqrt{\frac{\alpha'}{2}}\Big(\alpha_{-1} \cdot e \cdot k\tilde{c}_{-1} + (k \cdot \tilde{\beta})\tilde{c}_{-1}b_{-1} - (k \cdot \tilde{\gamma})c_{-1}\tilde{c}_{-1} + \beta \cdot \alpha_{-1}k \cdot \tilde{\alpha}_{-1} \\ &\quad + \tilde{B}_c k \cdot \tilde{\alpha}_{-1}c_{-1} + Bb_{-1}k \cdot \tilde{\alpha}_{-1}\Big)|0, 0; \mathbf{k}\rangle. \end{aligned} \tag{5.15}$$

This requires that,

$$e \cdot k = k \cdot e = 0, \tag{5.16}$$

$$B_{\tilde{c}} = \tilde{B}_c = 0, \tag{5.17}$$

$$k \cdot \gamma = k \cdot \tilde{\gamma} = 0, \tag{5.18}$$

$$\beta_\mu = \tilde{\beta}_\mu = 0, \tag{5.19}$$

$$B = 0. \tag{5.20}$$

The first pair of relations require that e be transverse, the second pair eliminate $B_{\tilde{c}}$ and $\tilde{B}_c$, while the last two ensure that all the remaining ghost modes (γ, $\tilde{\gamma}$, and C) have zero norm. The requirement that e be transverse leaves 24×24 transverse, spatially polarized, positive modes as well as modes of the form $e \sim \xi k$ and $e \sim k\xi$ which have zero norm (since $k^2 = 0$). Now, note that all these zero modes are exact - this can be seen by noting that they all arise in the above formula as Q_B or $\tilde{Q}_B$ of some state. Thus, our physical Hilbert space just consists of the remaining 24×24 positive modes, corresponding to the graviton, dilaton, and antisymmetric tensor.

3. Problem 10.13 of Polchinski, Vol. 2:

   We consider the operator products of the BRST current of the superstring (all products are normal ordered),

$$j_B = cT_B^{\mathrm{m}} + \gamma T_F^{\mathrm{m}} + bc\partial c + \frac{3}{4}(\partial c)\beta\gamma + \frac{1}{4}c(\partial\beta)\gamma - \frac{3}{4}c\beta(\partial\gamma) - b\gamma\gamma. \tag{5.21}$$

   Consider the operator product,

$$\begin{aligned} j_B(z)b(0) &\sim \frac{1}{z}T_B^{\mathrm{m}}(z) - \frac{1}{z^2}bc(z) - \frac{1}{z}b\partial c(z) - \frac{3}{4z^2}\beta\gamma + \frac{1}{4z}(\partial\beta)\gamma(z) - \frac{3}{4z}\beta\partial\gamma(z) \\ &\sim \ldots + \frac{1}{z}\left[T_B^{\mathrm{m}}(0) - (\partial b)c(0) - 2b\partial c(0) - \frac{1}{2}(\partial\beta)\gamma(0) - \frac{3}{2}\beta(\partial\gamma)(0)\right] \\ &= \ldots + \frac{1}{z}T_B(0), \end{aligned} \tag{5.22}$$

   and also the operator product (we note that Polchinski has a sign error in Eqn. 10.2.21b),

$$\begin{aligned} j_B(z)\beta(0) &\sim \frac{1}{z}T_F^{\mathrm{m}}(z) + \frac{3}{4z}(\partial c)\beta(z) + \frac{1}{4z}c(\partial\beta)(z) + \frac{3}{4z^2}c\beta(z) - \frac{2}{z}b\gamma(z) \\ &\sim \frac{1}{z}\left[T_F^{\mathrm{m}}(0) + \frac{3}{2}(\partial c)\beta(0) + c(\partial\beta)(0) - 2b\gamma(0)\right] \\ &= \ldots + \frac{1}{z}T_F(0). \end{aligned} \tag{5.23}$$

   By using these relations and the relation of the operator products to commutators of charges, we see that

$$\{Q_B, b_n\} = L_n, \qquad [Q_B, \beta_r] = G_r. \tag{5.24}$$

   Now, these facts are exactly what we need to go through precisely the same arguments as in problem 1 above to show that the BRST operator is nilpotent iff the total central charge vanishes, though in this case we need $c^{\mathrm{g}} = -15 = -c^{\mathrm{m}}$.