PHY 250 (P. Horava) Homework Assignment 6 Solutions
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1. Exercise 6.8 of Polchinski, Vol. 1:

We consider the tree-level amplitude for three massless vector bosons with momenta k_i, polarizations e_i and Chan-Paton factors λ^{a_i} in open bosonic string theory. This amplitude is given by ($g'_o = g_o(2\alpha')^{-1/2}$),

$$\begin{aligned} S(k_1,&a_1,e_1;k_2,a_2,e_2;k_3,a_3,e_3) = \\ &(-ig'_o)^3(g_o)^{-2}\left\langle {}^{\star}_{\star}\, c^1 e_1\cdot\dot{X}e^{ik_1\cdot X}(y_1)\, {}^{\star\star}_{\star\star}\, c^1 e_2\cdot\dot{X}e^{ik_2\cdot X}(y_2)\, {}^{\star\star}_{\star\star}\, c^1 e_3\cdot\dot{X}e^{ik_3\cdot X}(y_3)\, {}^{\star}_{\star}\right\rangle_{D_2} \\ &\times \mathrm{Tr}(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}) + (k_2,e_2,a_2 \leftrightarrow k_2,e_3,a_3). \end{aligned} \tag{6.1}$$

where the two terms correspond to the two different cyclic permutations of insertions on the boundary. To evaluate this, we first note that since $k_1+k_2+k_3=0$, and as $k_i^2=0$, we must have that $(k_i+k_j)^2 = k_l^2 = 0 = 2k_i\cdot k_j$, $i\neq j\neq l$. Further, as $e_i\cdot k_i = 0$, we have that $e_i\cdot k_j = e_i\cdot(-k_i-k_l) = -e_i\cdot k_l$, and so $e_i\cdot k_j = \frac{1}{2}e_i\cdot k_{jl}$, where $k_{jl}=k_j-k_l$. Using the general result, Polchinski 6.2.36, where the contractions between the $\dot{X}$'s and the $e^{ik\cdot X}$'s are explicitly written out, while the q^μ's represent the still to be carried out contractions between different $\dot{X}$'s (each of which yields $-2\alpha'(y-y')^{-2}\eta^{\mu\nu}$),

$$\begin{aligned} &\left\langle {}^{\star}_{\star}\, e_1\cdot\dot{X}e^{ik_1\cdot X}(y_1)\, {}^{\star\star}_{\star\star}\, e_2\cdot\dot{X}e^{ik_2\cdot X}(y_2)\, {}^{\star\star}_{\star\star}\, e_3\cdot\dot{X}e^{ik_3\cdot X}(y_3)\, {}^{\star}_{\star}\right\rangle_{D_2} \\ &\quad= iC^X_{D_2}(2\pi)^{26}\delta^{26}(\textstyle\sum k_i)|y_{12}|^{2\alpha' k_1\cdot k_2}|y_{13}|^{2\alpha' k_1\cdot k_3}|y_{23}|^{2\alpha' k_2\cdot k_3} \\ &\quad\times\left\langle\left[-2i\alpha'\left(\frac{e_1\cdot k_2}{y_{12}}+\frac{e_1\cdot k_3}{y_{13}}\right)+e_1\cdot q(y_1)\right]\left[-2i\alpha'\left(\frac{e_2\cdot k_1}{y_{21}}+\frac{e_2\cdot k_3}{y_{23}}\right)+e_2\cdot q(y_2)\right]\right. \\ &\quad\times\left.\left[-2i\alpha'\left(\frac{e_3\cdot k_1}{y_{31}}+\frac{e_3\cdot k_1}{y_{31}}\right)+e_3\cdot q(y_3)\right]\right\rangle_{D_2} \\ &\quad= iC^X_{D_2}(2\pi)^{26}\delta^{26}(\textstyle\sum k_i)\left((-i\alpha')(-2\alpha')\dfrac{e_1\cdot k_{23}e_2\cdot e_3+e_2\cdot k_{31}e_3\cdot e_1+e_3\cdot k_{12}e_1\cdot e_2}{y_{12}y_{13}y_{23}}\right. \\ &\quad\quad\left.+(-i\alpha')^3\dfrac{e_1\cdot k_{23}e_2\cdot k_{31}e_3\cdot k_{12}}{y_{12}y_{13}y_{23}}\right) \end{aligned} \tag{6.2}$$

Now, using the fact that (Polchinski 6.3.10 - the absolute value is inserted here since if arises as a FP determinant),

$$\langle c(y_1)c(y_2)c(y_3)\rangle = C^{\mathrm{g}}_{D_2}|y_{12}y_{23}y_{13}| \tag{6.3}$$

and the relation among the normalizations found by comparing the three and four point tachyon amplitudes (Polchinski 6.4.14),

$$C_{D_2} = e^{-\lambda}C^X_{D_2}C^{\mathrm{g}}_{D_2} = \frac{1}{\alpha' g_o^2}, \tag{6.4}$$

we easily obtain the final amplitude,

$$\begin{aligned} S(k_1,&a_1,e_1;k_2,a_2,e_2;k_3,a_3,e_3) = (-ig'_o)^3(g_o)^{-2}(iC^X_{D_2})(2i\alpha'^2)(C^{\mathrm{g}}_{D_2}|y_{12}y_{23}y_{13}|)(2\pi)^{26}\delta^{26}(\textstyle\sum k_i)\times \\ &\left(\frac{e_1\cdot k_{23}e_2\cdot e_3+e_2\cdot k_{31}e_3\cdot e_1+e_3\cdot k_{12}e_1\cdot e_2+\frac{\alpha'}{2}e_1\cdot k_{23}e_2\cdot k_{31}e_3\cdot k_{12}}{y_{12}y_{13}y_{23}}\mathrm{Tr}(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3})+(2\leftrightarrow 3)\right) \\ =&ig'_o(2\pi)^{26}\delta^{26}(\textstyle\sum k_i)\,(e_1\cdot k_{23}e_2\cdot e_3+e_2\cdot k_{31}e_3\cdot e_1+e_3\cdot k_{12}e_1\cdot e_2 \\ &\left.+\frac{\alpha'}{2}e_1\cdot k_{23}e_2\cdot k_{31}e_3\cdot k_{12}\right)Tr(\lambda^{a_1}[\lambda^{a_2},\lambda^{a_3}]). \end{aligned} \tag{6.5}$$

In the low energy limit, $\alpha' \to 0$, we see that the last term vanishes, and we are left only with terms linear in k. These are exactly the form of the terms which arise from the three boson vertex in Yang-Mills theory, Tr $A \wedge A \wedge *dA$.