HW 2 SOLUTIONS

Problem 1

HF Question 3, pg.11.

For each part we state the number of DOF's and briefly decribe them.

a) 1-the angle describing the rotation of the disk.

b) 3-normally a rigid body has six degrees of freedom but fixing a point on the body removes 3 of them (corresponding to the three coordinates needed to describe the position of the fixed point).

c) 2-the ground, or any surface, is 2-dimensional.

d) 5-two point particles have 6 total degrees of freedom but imposing that they keep a constant distance from each other provides one constraint.

f) 18- 6 for each rigid body.

Problem 2

HF Problem 6, pg.28

a) Adding a constant to the Lagrangian clearly doesn't change the equations of motion since the constant will drop out when we take $\frac{\partial L}{\partial q}$ and $\frac{\partial L}{\partial \dot{q}}$. Multiplying by a constant will just result in a multiplication of the E-L equations by that constant, which can then be canceled out.

b)The Euler-Lagrange equations will be unchanged under the addition of a term $\frac{dF}{dt}$ if and only if

$$\frac{d}{dt}\frac{\partial(\frac{dF}{dt})}{\partial \dot{q}_k} = \frac{\partial(\frac{dF}{dt})}{\partial q_k} \tag{1}$$

To verify (**??**), first note that $\frac{dF}{dt} = \frac{\partial F}{\partial q_k}\dot{q}_k + \frac{\partial F}{\partial t}$ (summation over repeated indices is implied), so

$$\begin{aligned} \frac{d}{dt}\frac{\partial(\frac{dF}{dt})}{\partial \dot{q}_k} &= \frac{d}{dt}[\frac{\partial}{\partial \dot{q}_k}(\frac{\partial F}{\partial q_j}\dot{q}_j + \frac{\partial F}{\partial t})] && (2) \\ &= \frac{d}{dt}\frac{\partial F}{\partial q_k} && (3) \end{aligned}$$

$$= \frac{\partial^2 F}{\partial q_k \partial q_j}\dot{q}_j + \frac{\partial^2 F}{\partial q_k \partial t} \quad (4)$$

while

$$\frac{\partial}{\partial q_k}\frac{dF}{dt} = \frac{\partial}{\partial q_k}(\frac{\partial F}{\partial q_k}\dot{q}_k + \frac{\partial F}{\partial t}) \quad (5)$$

$$= \frac{\partial^2 F}{\partial q_k \partial q_j}\dot{q}_j + \frac{\partial^2 F}{\partial q_k \partial t} \quad (6)$$

so we have the exact same expression for both sides of (**??**) and we're done.

Problem 3

HF Problem 12, pg. 30

We have a free particle, which by definition means $V = 0$ so we only have a kinetic term $T = \frac{1}{2}m(\dot{x^2} + \dot{y^2})$ in our lagrangian. Using $x = rcos\theta$ and $y = rsin\theta$ we have

$$\dot{x} = \dot{r}cos\theta - r\dot{\theta}sin\theta \quad (7)$$

$$\dot{y} = \dot{r}sin\theta + r\dot{\theta}cos\theta \quad (8)$$

so we find

$$T = \frac{1}{2}m(\dot{r^2} + r^2\dot{\theta^2}) \quad (9)$$

and so taking derivatives yields

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{r}} = m\ddot{r} \quad (10)$$

$$\frac{\partial T}{\partial r} = mr\dot{\theta^2} \quad (11)$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} = 2mr\dot{\theta}\dot{r} + mr^2\ddot{\theta} \quad (12)$$

$$\frac{\partial T}{\partial \theta} = 0. \quad (13)$$

Now, using Hand and Finch's "golden rule" $\mathcal{F}_k \equiv \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial q_k} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q_k}} - \frac{\partial T}{\partial q_k}$ (where $\mathbf{r}$ is the 2-dimensional Euclidean position vector of our particle) and the facts that $\frac{\partial \mathbf{r}}{\partial r} = \hat{\mathbf{e}}_r$ and $\frac{\partial \mathbf{r}}{\partial \theta} = r\hat{\mathbf{e}}_\theta$, we have

$$\mathbf{a_r} = \frac{1}{m}\mathbf{F} \cdot \hat{\mathbf{e}}_r = \frac{1}{m}\mathcal{F}_r = \ddot{r} - r\dot{\theta}^2 \tag{14}$$

$$\mathbf{a}_\theta = \frac{1}{m}\mathbf{F} \cdot \hat{\mathbf{e}}_\theta = \frac{1}{mr}\mathcal{F}_r = 2\dot{r}\dot{\theta} + r^2\ddot{\theta} \tag{15}$$

Problem 4

HF Problem 22, pg. 34

Taking (X, θ) as our generalized coordinates where X is defined in the problem and θ is the angle that the pendulum makes with the vertical, we have the following expressions for the Euclidean x, y coordinates of the box and plane in terms of (X, θ):

$$\begin{aligned} x_b &= X && (16) \\ y_b &= constant && (17) \\ x_p &= X + lsin\theta && (18) \\ y_p &= constant + l(1 - cos\theta) && (19) \\ & && (20) \end{aligned}$$

so taking time derivatives yields

$$\begin{aligned} x_b &= \dot{X} && (21) \\ y_b &= 0 && (22) \\ x_p &= \dot{X} + l\dot{\theta}cos\theta && (23) \\ y_p &= l\dot{\theta}sin\theta. && (24) \\ & && (25) \end{aligned}$$

Plugging in the above expressions into T and setting the gravitational potential to be 0 when the pendulum is hanging straight down, we have

$$L = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{X}^2 + 2l\dot{X}\dot{\theta}cos\theta + l^2\dot{\theta}^2) - mgl(1 - cos\theta). \tag{26}$$

Turning the crank then yields two (coupled) EOM's for X and θ respectively:

$$\begin{aligned} (M + m)\ddot{X} + ml\ddot{\theta}cos\theta - ml\dot{\theta}^2 sin\theta &= 0 && (27) \\ \ddot{X}lcos\theta + l^2\ddot{\theta} &= -glsin\theta. && (28) \end{aligned}$$

Problem 5

HF Problem 24, pg.34

We only need compute T, since $p_\theta \equiv \frac{\partial L}{\partial \dot{\theta}}$ and V doesn't depend on $\dot{\theta}$. Since we're dealing with a rigid body, $T = \frac{1}{2} I \omega^2$ where ω is the angular velocity and I is the moment of inertia about the axis of rotation. Clearly $\omega = \dot{\theta}$, and we have

$$\begin{aligned} I &= \int_0^{l-d} (m/l) x^2 dx + \int_0^d (m/l) x^2 dx && (29) \\ &= (m/3l)[(l-d)^3 + d^3] && (30) \end{aligned}$$

although we will not use this expression for I explicitly in our answer. Thus $T = \frac{1}{2} I \dot{\theta}^2$ so

$$p_\theta = I\dot{\theta} = I\omega \quad (31)$$

the angular momentum!