HW 5 SOLUTIONS

Problem 1

HF Chapter 3, problem 21d

Our EOM is

$$\ddot{q} + 2\dot{q} + q = \delta(t). \tag{1}$$

To solve, we take the general solution to the homogeneous equation $q(t) = Ae^{-t} + bte^{-t}$ for $t > 0$ and match it to $q(0) = 0$ for $t < 0$ using appropriate boundary conditions. Integrating the EOM from $-\epsilon$ to $+\epsilon$ and letting $\epsilon \to 0$ yields

$$\dot{q}(0^+) = 1 \tag{2}$$

which when combined with $q(0) = 0$ yields

$$-A + B = 1 \tag{3}$$
$$A = 0 \tag{4}$$

so

$$q(t) = \begin{cases} te^{-t} & t > 0 \\ 0 & t < 0 \end{cases} \quad (5)$$

Hence

$$G(t - t') = (t - t')e^{-(t-t')} \tag{6}$$

Alternatively, one can just take the derivative of the step function solution $q_{step}(t) = 1 - e^{-t} - te^{-t}$ since the derivative of a step function is a delta function.

Problem 2

HF chapter 3 problem 13

Our EOM is

$$\ddot{q} + q = sinat \tag{7}$$

so we try a particular solution of the form $q_p = Asinat$, yielding

$$-Aa^2 + A = 1 \Rightarrow A = \frac{1}{1 - a^2} \tag{8}$$

Thus the general solution is

$$q(t) = Asint + Bsint + \frac{1}{1-a^2} sinat \tag{9}$$

so applying our initial conditions $q(0) = \dot{q}(0) = 0$ yields (we skip the algebra)

$$q(t) = \frac{1}{a^2-1}(asint - sinat) \tag{10}$$

Problem 3

HF Chapter 3 problem 19

a) A DSHO with 0 inital velocity and position and step function driving force satisfies

$$q_0(t) = 1 - e^{-t/2Q}(cos\omega' t + \frac{1}{2Q\omega'} sin\omega' t) \tag{11}$$

where we have corrected the erroneus minus sign found in the text's eqn (3.48). In the case we're examining now, we have a driving function

$$F(t) = \theta(t + T/2) - \theta(t - T/2) \tag{12}$$

so we suspect that we may solve this for $t > T/2$ by taking a superposition of appropriately translated versions of (11), i.e.

$$q(t) = \begin{cases} 0 & t < -T/2 \\ q_0(t + T/2) & -T/2 < t < T/2 \\ q_0(t + T/2) - q_0(t - T/2) & t > T/2 \end{cases} \tag{13}$$

Our guess for $-T/2 < t < T/2$ makes sense since it is the solution for a step function driving force at $t = -T/2$. Our guess for $t > T/2$ makes sense since we are then accounting for both steps. To check that we have really written down the correct solution, we note that our $q(t)$ satisfies the EOM for all t, and that

$$lim_{t\to T/2^-} q(t) = q_0(t + T/2)|_{t=T/2} = q_0(T) = lim_{t\to T/2^+} q(t) \tag{14}$$

since $q_0(t - T/2)|_{t=T/2} = q_0(0) = 0$ so $q(t)$ is continuous at $t = T/2$. Similarly, one uses $\dot{q}_0(0) = 0$ to show that $\dot{q}$ is continuous at $t = T/2$. Hence our solution is the correct one, and plugging in (11) yields the text's eqn (3.103) except for an overall minus sign, yet another error in the text.

b)From the EOM

$$\ddot{q} + 2\dot{q} + q = \theta(t + T/2) - \theta(t - T/2) \tag{15}$$

we see that, since q and $\dot{q}$ are continuous, the discontinuity in $\ddot{q}$ at $t = -T/2$ comes entirely from the step function and has height 1.

c)For T small, $q(t \pm T/2) \sim q(t) \pm \frac{T}{2}\dot{q}(t)$ so

$$\begin{aligned}
& q(t + T/2) - q(t - T/2) \sim T\dot{q}(t) && (16)\\
= \quad & -Te^{-t/2Q}[-\tfrac{1}{2Q}(cos\omega' t + \tfrac{1}{2Q\omega'}sin\omega' t) + (-\omega' sin\omega' t + \tfrac{1}{2Q}cos\omega' t)] && (17)\\
= \quad & \tfrac{T}{\omega'}e^{-t/2Q}(\tfrac{1}{4Q^2} + \omega'^2)sin\omega' t && (18)\\
= \quad & \tfrac{T}{\omega'}e^{-t/2Q}sin\omega' t && (19)\\
= \quad & G(t) && (20)
\end{aligned}$$

where we used $\omega'^2 = 1 - \frac{1}{4Q^2}$.

Problem 4

HF Chapter 3 problem 22

The Lagrangian

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{4}q^4 \tag{21}$$

yields the desired EOM. The energy is

$$E = \frac{1}{2}\dot{q}^2 + \frac{1}{4}q^4 \tag{22}$$

and it is conserved since we have no explicit time-dependence in the Lagrangian. The oscillator is said to be nonlinear since the EOM contains terms which are nonlinear (i.e. not first order in q, $\dot{q}$ and $\ddot{q}$).An easily verified consequence of this is that the set of solutions to the EOM do not form a vector space (i.e. one cannot superimpose solutions).