HW 6 SOLUTIONS

Problem 1

HF Chapter 4, problem 24

Our effective potential here is

$$V_{\text{eff}} = -\frac{k}{r} + \frac{\beta}{r^2} + \frac{\beta_0}{r^2} \tag{1}$$

where $\beta_0 \equiv \frac{l^2}{2m}$. We can write this as

$$V_{\text{eff}} = -\frac{k}{r} + \frac{\beta_0}{r^2}\alpha^2, \quad \alpha^2 \equiv 1 + \frac{\beta}{\beta_0} \tag{2}$$

Now, we would like to get rid of the α in (2) to obtain the EOM for the Kepler problem, whose solution we know. To do this, define $\phi' \equiv \alpha\phi$, so that

$$\beta_0\alpha^2 = \frac{(mr^2\dot{\phi})^2}{2m}\alpha^2 = \frac{(mr^2\dot{\phi}')^2}{2m} \equiv \frac{l'^2}{2m} \tag{3}$$

Then we have the Kepler problem in terms of ϕ', l' so we just read off the solution:

$$\begin{aligned} \frac{p}{r} &= 1 + \epsilon cos\phi', \quad p \equiv \frac{l'^2}{mk} = \frac{l^2\alpha^2}{mk} && (4) \\ &= 1 + \epsilon cos(\alpha\phi) && (5) \end{aligned}$$

So our definition of α agrees with the text's, and we have found it equal to $\sqrt{1 + \frac{\beta}{\beta_0}}$.

Now, according to (5), r attains it's minimum (perihelion) at $\phi = 0, \frac{2\pi}{\alpha}$, so the precession angle $\Delta\phi$ after one orbit, which is the difference between 2π and the actual angular distance traveled between perihelia, is

$$\Delta\phi = 2\pi - \frac{2\pi}{\alpha} = \frac{2\pi}{\alpha}(\alpha - 1) \tag{6}$$

The time it takes to precess this much is the period, given by eqn (4.60),

$$\tau = 2\pi\sqrt{m/k}\left(\frac{p}{1-\epsilon^2}\right)^{3/2} = 2\pi\frac{(\alpha l)^3}{mk^2(1-\epsilon^2)^{3/2}} \tag{7}$$

so the rate of precession is

$$\omega_{pre} = \frac{\Delta\phi}{\tau} = \frac{\alpha - 1}{\alpha^4}\frac{mk^2(1-\epsilon^2)^{3/2}}{l^3}. \tag{8}$$

Problem 2

a)The equation for the hyperbolic orbits in cartesian coordinates is

$$p = -\sqrt{x^2 + y^2} + \epsilon x \tag{9}$$

which can be rearranged to give

$$\frac{y}{x} = \pm\sqrt{-1 + (\frac{p}{x} - \epsilon)^2}. \tag{10}$$

We are looking for the straight line which is the limit of this curve as x (and y) go to infinity. Substituting $y = mx + b$ into (10) and taking the limit as x goes to infinity yields

$$m = \pm\sqrt{\epsilon^2 - 1} \tag{11}$$

To find the intercept we demand that the difference between our line $y = \pm\sqrt{\epsilon^2 - 1}x + b$ and the hyperbola $y = \pm x\sqrt{-1 + (\frac{p}{x} - \epsilon)^2}$ go to 0 as x approaches infinity. We Taylor expand this difference in the small quantity $\frac{1}{x}$:

$$\begin{aligned} 0 &= \pm\sqrt{\epsilon^2 - 1}x + b \mp x\sqrt{-1 + (\frac{p}{x} - \epsilon)^2} && (12) \\ &= \pm\sqrt{\epsilon^2 - 1}x + b \mp x\sqrt{-1 + \tfrac{p^2}{x^2} - 2\epsilon\tfrac{p}{x} + \epsilon^2} && (13) \\ &= \pm\sqrt{\epsilon^2 - 1}x + b \mp x\sqrt{\epsilon^2 - 1}\sqrt{1 + \tfrac{1}{\epsilon^2-1}(\tfrac{p^2}{x^2} - 2\epsilon\tfrac{p}{x})} && (14) \\ &\sim \pm\sqrt{\epsilon^2 - 1}x + b \mp x\sqrt{\epsilon^2 - 1}(1 + \tfrac{1}{2}\tfrac{1}{\epsilon^2-1}(\tfrac{p^2}{x^2} - 2\epsilon\tfrac{p}{x})) && (15) \\ &= b \pm \tfrac{\epsilon p}{\sqrt{\epsilon^2-1}} + O(\tfrac{1}{x}) && (16) \end{aligned}$$

so taking the limit as x goes to infinity yields

$$b = \mp\frac{\epsilon p}{\sqrt{\epsilon^2 - 1}} \tag{17}$$

so our line is

$$y = \pm\sqrt{\epsilon^2 - 1}x \mp \frac{\epsilon p}{\sqrt{\epsilon^2 - 1}}. \tag{18}$$

b)At infinity, the actual hyperbolic orbit of the particle and the asymptotes coincide, so we can evaluate the angular momentum l at infinity to get

$$l = |\mathbf{r}_\infty \times \mathbf{p}_\infty| = \mu r_\infty v_\infty sin\theta = \mu b v_\infty \tag{19}$$

where we used elementary trigonometry to see that $b = rsin\theta$ for the asymptote everywhere, and hence for the hyperbola at infinity.

Then, using $v_\infty = \sqrt{2E/\mu}$ and $l = \sqrt{\frac{\mu k^2}{2E}(\epsilon^2 - 1)}$ we have

$$b = \frac{l}{\mu v_\infty} = \frac{k}{2E}\sqrt{\epsilon^2 - 1} \tag{20}$$

Problem 3

a)We compute $\frac{d\mathbf{A}}{dt}$ in cylindrical basis vectors:

$$\begin{aligned}
\frac{d\mathbf{A}}{dt} &= \frac{d\mathbf{p}}{dt} \times \mathbf{L} - \mu k \frac{d\hat{\mathbf{r}}}{dt} && (21)\\
&= -\frac{k}{r^2}\hat{\mathbf{r}} \times l\hat{\mathbf{z}} - \mu k \dot{\theta}\hat{\theta} && (22)\\
&= k(\frac{l}{r^2} - \mu\dot{\theta})\hat{\theta} && (23)\\
&= 0. && (24)
\end{aligned}$$

where in the first equality we used the fact that $\mathbf{L}$ is constant, in the second equality we used $\frac{d\mathbf{p}}{dt} = -\frac{k}{r^2}\hat{\mathbf{r}}$ and $\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\hat{\theta}$, and in the last equality we used the definition of l.

b)Assume that perihelion (closest approach) occurs along the positive x-axis. Then, using that at perihelion $\mathbf{p} = l/r\mathbf{y}$, we evaluate $\mathbf{A}$:

$$\mathbf{A} = \frac{l}{r}\mathbf{y} \times l\mathbf{z} - \mu k\mathbf{x} = \mu k(\frac{l^2}{\mu k r} - 1)\mathbf{x} = \mu k\epsilon\mathbf{x} \tag{25}$$

so A has magnitude $\mu k\epsilon$ and points in the direction of the perihelion. It's existence is thus seen to be linked with the fact that the oribts in the Kepler problem are closed, which is due to an extra symmetry peculiar to the Kepler problem (and the isotropic harmonic oscillator, the other system with closed orbits).

Problem 4

a)Using $\mathbf{r} = (Rcos\theta, Rsin\theta, z)$ we have

$$L = \frac{m}{2}\dot{\mathbf{r}}^2 + \frac{k}{2}\mathbf{r}^2 = \frac{m}{2}(R^2\dot{\theta}^2 + \dot{z}^2) + \frac{k}{2}(R^2 + z^2) \tag{26}$$

b)We have

$$\frac{\partial L}{\partial \dot{z}} = p_z = m\dot{z} \quad \Rightarrow \quad \dot{z} = \frac{p_z}{m} \tag{27}$$

$$\frac{\partial L}{\partial \dot{\theta}} = p_\theta = mR^2\dot{\theta} \quad \Rightarrow \quad \dot{\theta} = \frac{p_\theta}{mR^2} \tag{28}$$

so

$$H = H(z, \theta, p_z, p_\theta) = \frac{p_z^2}{2m} + \frac{p_\theta^2}{2mR^2} + \frac{k}{2}(R^2 + z^2) \tag{29}$$

and so Hamilton's equations are

$$\dot{z} = \frac{p_z}{m} \tag{30}$$

$$\dot{\theta} = \frac{p_\theta}{mR^2} \tag{31}$$

$$\dot{p_z} = -kz \tag{32}$$

$$\dot{p_\theta} = 0 \tag{33}$$

Problem 5

a)Using $\sqrt{1-x} \sim 1 - \frac{1}{2}x$ gives

$$L \approx -mc^2 + \frac{m}{2}\dot{\mathbf{x}}^2 - e\Phi + \frac{e}{c}\dot{\mathbf{x}} \cdot \mathbf{A} \tag{34}$$

which up to the irrelevant constant term $-mc^2$ (which represents the rest mass) is the same as our nonrelativistic lagrangian.

b)We have

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = \frac{m\dot{x}^i}{\sqrt{1 - \frac{\dot{\mathbf{x}}^2}{c^2}}} + \frac{e}{c}A_i \tag{35}$$

so in terms of vectors

$$\frac{m\dot{\mathbf{x}}}{\sqrt{1 - \frac{\dot{\mathbf{x}}^2}{c^2}}} = \mathbf{p} - \frac{e}{c}\mathbf{A}. \tag{36}$$

Dotting (36) with itself yields

$$(\mathbf{p} - \frac{e}{c}\mathbf{A})^2 = \frac{m^2\dot{\mathbf{x}}^2}{1 - \dot{\mathbf{x}}^2}$$

which can be solved for $\dot{\mathbf{x}}^2$, yielding

$$\dot{\mathbf{x}}^2 = \frac{c^2(\mathbf{p} - \frac{e}{c}\mathbf{A})^2}{m^2c^2 + (\mathbf{p} - \frac{e}{c}\mathbf{A})^2}. \tag{38}$$

Combining (36) and (38) yields $\dot{\mathbf{x}}$ in terms of $\mathbf{p}$:

$$\begin{aligned}
\dot{\mathbf{x}} &= \frac{\sqrt{1 - \frac{\dot{\mathbf{x}}^2}{c^2}}}{m}(\mathbf{p} - \frac{e}{c}\mathbf{A}) && (39)\\
&= \frac{1}{m}\sqrt{1 - \frac{(\mathbf{p} - \frac{e}{c}\mathbf{A})^2}{m^2c^2 + (\mathbf{p} - \frac{e}{c}\mathbf{A})^2}}(\mathbf{p} - \frac{e}{c}\mathbf{A}) && (40)\\
&= \frac{1}{m}\sqrt{\frac{m^2c^2}{m^2c^2 + (\mathbf{p} - \frac{e}{c}\mathbf{A})^2}}(\mathbf{p} - \frac{e}{c}\mathbf{A}) && (41)
\end{aligned}$$

so we can now write the Hamiltonian, substituting our admittedly unwieldy expression (41) for $\dot{\mathbf{x}}$:

$$\begin{aligned}
H &= \mathbf{p} \cdot \dot{\mathbf{x}} - L && (42)\\
&= \sqrt{\frac{c^2}{m^2c^2 + (\mathbf{p} - \frac{e}{c}\mathbf{A})^2}}(\mathbf{p}^2 - \frac{e}{c}\mathbf{A} \cdot \mathbf{p}) + mc^2\sqrt{\frac{m^2c^2}{m^2c^2 + (\mathbf{p} - \frac{e}{c}\mathbf{A})^2}} + e\Phi - \\
&\quad \frac{e}{mc}\sqrt{\frac{m^2c^2}{m^2c^2 + (\mathbf{p} - \frac{e}{c}\mathbf{A})^2}}(\mathbf{A} \cdot \mathbf{p} - \frac{e}{c}\mathbf{A}^2) && (43)\\
&= \frac{1}{\sqrt{m^2c^2 + (\mathbf{p} - \frac{e}{c}\mathbf{A})^2}}[c(\mathbf{p} - \frac{e}{c}\mathbf{A})^2 + m^2c^3] + e\Phi && (44)\\
&= \sqrt{m^2c^4 + c^2(\mathbf{p} - \frac{e}{c}\mathbf{A})^2} + e\Phi && (45)
\end{aligned}$$

where we have skipped a few steps of algebraic manipulation between equalities.

Now we wish to find Hamilton's equations. This will be ugly. We'll do this in vector notation (i.e. we'll write $\frac{\partial H}{\partial \mathbf{x}}$ to stand for the vector whose ith component is $\frac{\partial H}{\partial x^i}$), we'll make use of the vector identity $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A} \leftrightarrow \mathbf{B}$ as well as Maxwell's Equations $\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}$, $\mathbf{B} = \nabla\times\mathbf{A}$, and we'll skip some algebra. We have

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{x}} \tag{46}$$

$$= \quad -\frac{c^2}{2}\frac{\nabla(-\frac{2e}{c}\mathbf{p}\cdot\mathbf{A}+\frac{e^2}{c^2}\mathbf{A}^2)}{\sqrt{m^2c^4+c^2(\mathbf{p}-\frac{e}{c}\mathbf{A})^2}} - e\nabla\Phi \tag{47}$$

$$= \quad \frac{c^2[\frac{2e}{c}(\mathbf{p}\times\mathbf{B}+(\mathbf{p}\cdot\nabla)\mathbf{A}) - 2\frac{e^2}{c^2}(\mathbf{A}\times\mathbf{B}+(\mathbf{A}\cdot\nabla)\mathbf{A})]}{\sqrt{m^2c^4+c^2(\mathbf{p}-\frac{e}{c}\mathbf{A})^2}} - e\nabla\Phi \tag{48}$$

and now that we're done taking derivatives we can eliminate $\mathbf{p}$ for $\dot{\mathbf{x}}$, yielding (eventually)

$$\dot{\mathbf{p}} = \frac{[\frac{2me}{c}(\dot{\mathbf{x}}\times\mathbf{B}+(\dot{\mathbf{x}}\cdot\nabla)\mathbf{A})]}{2m} - e\nabla\Phi \tag{49}$$

which, when combined with

$$\dot{\mathbf{p}} = \frac{m\mathbf{a}}{\sqrt{1-\frac{\dot{\mathbf{x}}^2}{c^2}}} + \frac{m}{c^2(\sqrt{1-\frac{\dot{\mathbf{x}}^2}{c^2}})^3}(\dot{\mathbf{x}}\cdot\mathbf{a})\dot{\mathbf{x}} + \frac{e}{c}(\frac{\partial\mathbf{A}}{\partial t}+(\dot{\mathbf{x}}\cdot\nabla)\mathbf{A}) \tag{50}$$

(which is obtained by time differentiating (36)), yields, eventually,

$$\frac{m}{\sqrt{1-\frac{\dot{\mathbf{x}}^2}{c^2}}}\mathbf{a} + \frac{m}{c^2(\sqrt{1-\frac{\dot{\mathbf{x}}^2}{c^2}})^3}(\dot{\mathbf{x}}\cdot\mathbf{a})\dot{\mathbf{x}} = e(\mathbf{E}+\frac{1}{c}\dot{\mathbf{x}}\times\mathbf{B}) \tag{51}$$

which is the Lorenz force law plus an additional correction term which basically functions to prevent the mass from accelerating beyond the speed of light.