HW 9 SOLUTIONS

Problem 1

HF Chapter 6, problem 17

a) The Hamilton-Jacobi eqn. is

$$H\left(q, p = \frac{\partial S}{\partial q}\right) + \frac{\partial S}{\partial t} = 0 \tag{1}$$

or

$$\frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}\omega^2 q^2 + \frac{\partial S}{\partial t} = 0. \tag{2}$$

Setting

$$S(q, P \equiv \alpha, t) = W(q, \alpha) - \alpha t \tag{3}$$

yields

$$\frac{1}{2}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{1}{2}\omega^2 q^2 - \alpha = 0 \tag{4}$$

which can be rearranged to give

$$\frac{\partial W}{\partial q} = \sqrt{2\alpha - \omega^2 q^2} \tag{5}$$

or

$$W(q, \alpha) = \int \sqrt{2\alpha - \omega^2 q^2} dq. \tag{6}$$

b) Differentiating (6) with respect to α yields

$$\frac{\partial W}{\partial \alpha} = \int \frac{1}{\sqrt{2\alpha - \omega^2 q^2}} dq = \frac{1}{\omega} sin^{-1}\left(\frac{q\omega}{\sqrt{2\alpha}}\right) \tag{7}$$

where in integrating we made the substitution $q' = \frac{wq}{\sqrt{2\alpha}}$. Thus

$$\beta = \frac{\partial S}{\partial \alpha} = -t + \frac{\partial W}{\partial \alpha} = -t + \frac{1}{\omega} sin^{-1}\left(\frac{q\omega}{\sqrt{2\alpha}}\right) \tag{8}$$

which can be solved to give

$$q(t) = \frac{\sqrt{2\alpha}}{\omega} sin(\omega(t + \beta)) \tag{9}$$

c) From (1) and (3), we have

$$E = H = -\frac{\partial S}{\partial t} = \alpha \tag{10}$$

Problem 3

HF Chapter 6, Problem 18

For a particle of given total energy E, we have turning points when

$$E = V(q_{tp}) = Utan^2(aq_{tp}) \tag{11}$$

which can be inverted to give

$$q_{tp} = \pm\frac{1}{a}tan^{-1}\left(\sqrt{\frac{E}{U}}\right). \tag{12}$$

From

$$\frac{p^2}{2m} + V(q) = E \tag{13}$$

we have

$$p = \pm\sqrt{2(E - Utan^2(aq))} \tag{14}$$

so then the action is

$$\frac{1}{2\pi}\oint p\, dq = \frac{1}{\pi}\int_{-\frac{1}{a}tan^{-1}(\sqrt{\frac{E}{U}})}^{+\frac{1}{a}tan^{-1}(\sqrt{\frac{E}{U}})} \sqrt{2(E - Utan^2(aq))}\, dq \tag{15}$$

which is a bit tricky to integrate but which Mathematica eventually tells me is

$$I = \frac{\sqrt{2}}{a}(\sqrt{E+U} - \sqrt{U}). \tag{16}$$

Solving this for E yields

$$E(I) = H(I) = aI(\sqrt{2U} + \frac{aI}{2}) \tag{17}$$

so

$$\omega = \frac{\partial H}{\partial I} = a^2 I + a\sqrt{2U} = a^2[\frac{\sqrt{2}}{a}(\sqrt{E+U} - \sqrt{U})] + a\sqrt{2U} = \sqrt{2}a\sqrt{E+U} \tag{18}$$

which is the desired relationship.

Problem 4

HF Chapter 6 Problem 20a

The phase portrait is a rectangle in phase space of length d and height $2p = 2\sqrt{2E}$ so the action, being just the area in phase space enclosed by the orbit divided by 2π, is just

$$I = \frac{1}{2\pi} 2\sqrt{2E} d = \frac{\sqrt{2E} d}{\pi}. \tag{19}$$

Solving this for E yields

$$E = \left(\frac{I\pi}{\sqrt{2}D} \right)^2 \tag{20}$$

so

$$\omega = \frac{\partial E}{\partial I} = \frac{I\pi^2}{d^2} = \frac{\pi}{d}\sqrt{2E}. \tag{21}$$

Calculating the frequency in the naive way,

$$\omega = \frac{2\pi}{T} = 2\pi \frac{p}{2d} = \frac{\pi}{d}\sqrt{2E} \tag{22}$$

which agrees with (21).

Problem 5

HF Chapter 7 Problem 5

a) If K' is our rotating frame and K our inertial one, then we have

$$\frac{d\vec{\sigma}}{dt}|_{K'} = \frac{d\vec{\sigma}}{dt}|_{K} - \vec{\omega} \times \vec{\sigma} = g'(\vec{\sigma} \times \vec{B}) - \vec{\omega} \times \vec{\sigma} \tag{23}$$

which vanishes if we choose

$$\vec{\omega}_0 = -g'\vec{B}. \tag{24}$$