Physics 209: Classical Electromagnetism
Final Exam
Due by 12 noon, Dec 14, 2006
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Directions

- Please try solving both problems below.
- Please explain your solution.
- Even if you do not reach a final answer, please explain your plan for the solution and please write down the relevant formulas. That may help for partial credit.
- No need to rederive standard expressions that we derived in the classroom or that appear in the textbook.
- You may use your favorite software such as `Mathematica` or `Maple` for algebraic manipulations and integrations (but you don't need to!).
- **Please return your solutions to me by Thursday 12/14, 12 noon.**
- When you are finished, you can either return a handwritten solution to my office 403 LeConte Hall (slide it under the door if there is no one there), or email the solutions to

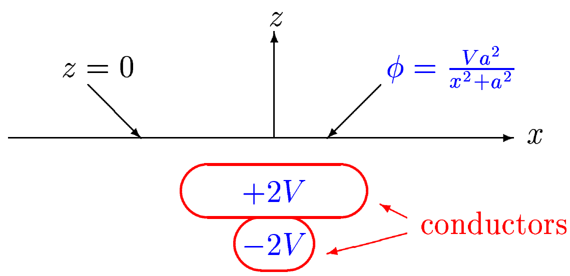
  `origa@socrates.berkeley.edu`

  I can read `TeX, LaTeX, MSWord` and `PDF`. (Please don't send Mathematica notebook files.) If you choose to return a handwritten solution, it would be good if you could also email me a message so as to be sure that I got your solution.
- During the exam period, you can communicate with me via e-mail.
- The maximal number of points that you can get for each problem is indicated in brackets $[\cdots]$. These numbers are *tentative* and might change slightly.

Good luck!

Problems

Problem 1: Electrostatics [45pts]



You are in charge of designing an experiment that requires a static potential of

$$\phi(x,y) = \frac{Va^2}{x^2+a^2}, \qquad V \text{ and } a \text{ are constants}$$

along the $x-y$ plane $z=0$. You decide to do this by designing a pair of appropriately shaped conductors to be placed somewhere below the plane $z=0$, such that one will be kept at a constant potential $+2V$ and the other will be kept at a constant potential $-2V$. (See the figure, but note that the shapes are not the correct ones and do not depict the actual answer!) Given V, what shapes will you design for the (surface of) the conductors and where will you place them? (Everything is in vacuum.)

Note: Don't be surprised if you find that the two conductors need to touch at some point. You may assume that the surface of each conductor is coated with a thin but good insulator, and don't worry about electrical breakdown.

Solution

This is a Dirichlet boundary value problem. The Green's function for the space $z > 0$ with Dirichlet boundary conditions on $z = 0$ is

$$G_d(x, y, z; x', y', z') = \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}}$$

$$\Phi(\vec{r}) = -\frac{1}{4\pi}\oint_S \Phi(\vec{r}')\frac{\partial G_D(\vec{r}, \vec{r}')}{\partial n'} da'.$$

In our case

$$\frac{\partial G_D(\vec{r}, \vec{r}')}{\partial n'} = -\frac{\partial G_D(\vec{r}, \vec{r}')}{\partial z'} = \frac{z' - z}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}^3} - \frac{z' + z}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}^3}$$

On the surface of the conductor $z' = 0$, and this becomes

$$\left.\frac{\partial G_D(\vec{r}, \vec{r}')}{\partial n'}\right|_{z'=0} = \frac{-2z}{\sqrt{(x-x')^2 + (y-y')^2 + z^2}^3}$$

Therefore for $z > 0$ we find

$$\begin{aligned}
\phi(x, y, z) &= \frac{z}{2\pi}\int \frac{\phi(x', y', 0)dx'dy'}{\sqrt{(x-x')^2 + (y-y')^2 + z^2}^3} = \frac{Va^2 z}{2\pi}\int \frac{dx'dy'}{(x'^2 + a^2)\sqrt{(x-x')^2 + (y-y')^2 + z^2}^3} \\
&= \frac{Va^2 z}{\pi}\int \frac{dx'}{(x'^2 + a^2)((x-x')^2 + z^2)} = \frac{Va(z+a)}{x^2 + (a+z)^2}.
\end{aligned}$$

The last integration can be done by a contour integration. Note also that if we define a complex coordinate $\zeta \equiv x + iz$ then

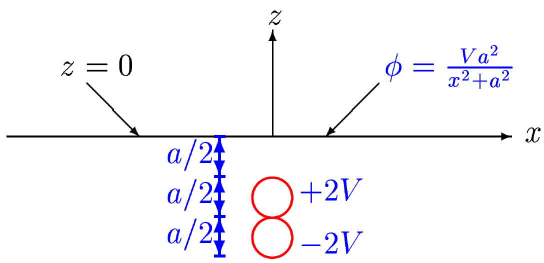
$$\phi = \mathrm{Re}\frac{iVa}{x + iz + ia} = \mathrm{Re}\frac{iVa}{\zeta + ia}.$$

So, it is the real part of an analytic function in ζ, and hence a solution to Laplace's equation in 2D.

Now we can find the position of the conductors. We analytically continue the potential to $z < 0$ and solve

$$\pm 2V = \frac{Va(z+a)}{x^2 + (a+z)^2} \Longrightarrow x^2 + (a+z)^2 = \pm\frac{1}{2}a(z+a) \Longrightarrow x^2 + (a+z \mp \frac{a}{4})^2 = \left(\frac{a}{4}\right)^2.$$

These are cylinders of radius $a/4$ with centers $z = -3a/4$ and $z = -5a/4$. They touch at $z = -a$.



Note 1

Some of you noticed that the potential can be generated by a straight wire of dipoles located at $(x, z) = (0, -a)$, with electric dipole moment per unit y-length given by

$$\frac{d\vec{p}}{dy} = 2\pi\epsilon_0 V a\hat{z}.$$

This can be seen by calculating the potential of a wire with constant charge density σ (using Gauss's law):

$$\Phi = -\frac{\sigma}{2\pi\epsilon_0}\ln\frac{r}{r_0}$$

(where r_0 is an arbitrary constant of integration), and taking the limit of two close and parallel wires with opposite charge densities.

Problem 2: Radiation [55pts]

In this problem the space above the $x-y$ plane ($z > 0$) is vacuum, and the space below the $x-y$ plane ($z < 0$) is filled with an insulating dielectric material with electric permittivity $\epsilon(\omega)$ (that depends on the frequency $\omega/2\pi$) and magnetic permeability $\mu = \mu_0$.

A thin wire of uniform charge density (per length) σ is placed below the plane $z = 0$ at a distance a from it and parallel to it, and is moving with constant velocity v along the $\hat{x}$-axis, so that it occupies the spacetime events $(ct, vt, y, -a)$ where $-\infty < y < \infty$ is the y-coordinate, and $-\infty < t < \infty$ is time. σ is the charge density in the lab frame, and the velocity v is larger than $c/\epsilon(\omega)$ for at least some range of ω's.

Find an expression for the power per unit length of (Cherenkov-like) radiation emitted by the wire.

Solution

We work with phasors. The charge density is

$$\rho(x, y, z) = \frac{1}{2\pi} \int \sigma\delta(z - a)\delta(x - vt)e^{i\omega t} dt = \frac{\sigma}{2\pi v}\delta(z - a)e^{i\frac{\omega}{v}x}.$$

The current density is $\vec{J} = \rho v\hat{x}$. We need to solve for the fields $\vec{E}$ and $\vec{B}$. Since the charge and current density are independent of y, we can look for a solution where the fields only depend on x, z. Since the charge and current density depend on x only through the factor $\exp(i\omega x/v)$, we can look for a solution where all the fields depend on x only through this factor.

The configuration is invariant under reflection $y \to -y$, whereas the electric field components transform as

$$E_x(x, y, z) \to E_x(x, -y, z), \qquad E_y(x, y, z) \to -E_y(x, -y, z), \qquad E_z(x, y, z) \to E_z(x, -y, z).$$

Since the fields are independent of y we must have $E_y = 0$. The magnetic fields transform as

$$B_x(x, y, z) \to -B_x(x, -y, z), \qquad B_y(x, y, z) \to B_y(x, -y, z), \qquad B_z(x, y, z) \to -B_z(x, -y, z).$$

So we conclude that $B_x = B_z = 0$. We can arrive at the same conclusion also by noting that the vector potential only has an $\hat{x}$-component and is independent of y.

So now we are looking for a solution of the form

$$\vec{E}(x, z) = e^{i\frac{\omega}{v}x}(E_x(z)\hat{x} + E_z(z)\hat{z}), \qquad \vec{B}(x, z) = e^{i\frac{\omega}{v}x}B_y(z)\hat{y}.$$

For $z < 0$, Maxwell's equations imply:

$$\begin{aligned}
0 &= \vec{\nabla} \cdot \vec{E} = (i\frac{\omega}{v}E_x + E_z')e^{i\frac{\omega}{v}x}, \\
0 &= \vec{\nabla} \times \vec{E} - i\omega\vec{B} = (E_x' - i\frac{\omega}{v}E_z - i\omega B_y)e^{i\frac{\omega}{v}x}\hat{y}, \\
0 &= \vec{\nabla} \times \vec{B} + i\omega\mu_0\epsilon\vec{E} = (-B_y' + i\omega\mu_0\epsilon E_x)e^{i\frac{\omega}{v}x}\hat{x} + (i\frac{\omega}{v}B_y + i\omega\mu_0\epsilon E_z)e^{i\frac{\omega}{v}x}\hat{z},
\end{aligned}$$

Looking at these equations component by component, we find four relations:

$$\begin{aligned}
E_x &= i\frac{v}{\omega}E_z', \\
B_y &= -\frac{i}{\omega}E_x' - \frac{1}{v}E_z = \frac{v}{\omega^2}E_z'' - \frac{1}{v}E_z, \\
B_y &= -v\mu_0\epsilon E_z, \\
E_x &= -\frac{i}{\omega\mu_0\epsilon}B_y' = \frac{iv}{\omega}E_z',
\end{aligned}$$

The last and the first equations are identical, and substituting the third equation into the second we get:

$$0 = E_z'' - \omega^2(\frac{1}{v^2} - \mu_0\epsilon)E_z.$$

We should keep only the outgoing wave solution (the one going down), so

$$E_z = De^{-ikz},$$

where the wavenumber is defined as

$$k = \omega\sqrt{\mu_0\epsilon - \frac{1}{v^2}}.$$

k is real only if $v \geq 1/\sqrt{\mu_0\epsilon}$ which is to say that v is greater than the speed of light in the medium.

For $(z > 0)$ we have

$$\begin{aligned}
0 &= \vec{\nabla}\cdot\vec{E} - \frac{1}{\epsilon_0}\rho = (i\frac{\omega}{v}E_x + E_z' - \frac{1}{\epsilon_0}\frac{\sigma}{2\pi v}\delta(z-a))e^{i\frac{\omega}{v}x}, \\
0 &= \vec{\nabla}\times\vec{E} - i\omega\vec{B} = (E_x' - i\frac{\omega}{v}E_z - i\omega B_y)e^{i\frac{\omega}{v}x}\hat{y}, \\
0 &= \vec{\nabla}\times\vec{B} + i\frac{\omega}{c^2}\vec{E} - \mu_0\vec{J} \\
&= (-B_y' + i\frac{\omega}{c^2}E_x - \frac{1}{2\pi}\mu_0\sigma\delta(z-a))e^{i\frac{\omega}{v}x}\hat{x} + (i\frac{\omega}{v}B_y + i\frac{\omega}{c^2}E_z)e^{i\frac{\omega}{v}x}\hat{z},
\end{aligned}$$

Looking at these equations component by component we get:

$$E_x = i\frac{v}{\omega}E_z' - \frac{i\sigma}{2\pi\epsilon_0\omega}\delta(z-a)$$

$$
\begin{aligned}
B_y &= -\frac{i}{\omega}E_x' - \frac{1}{v}E_z = \frac{v}{\omega^2}E_z'' - \frac{1}{v}E_z - \frac{\sigma}{2\pi\epsilon_0\omega^2}\delta'(z-a) \\
B_y &= -\frac{v}{c^2}E_z, \\
E_x &= -\frac{ic^2}{\omega}B_y' - \frac{ic^2\mu_0\sigma}{2\pi\omega}\delta(z-a) = \frac{iv}{\omega}E_z' - \frac{i\sigma}{2\pi\epsilon_0\omega}\delta(z-a)
\end{aligned}
$$

Eliminating B_y we find an ODE for $E_z(z)$:

$$
-\frac{v}{c^2}E_z = \frac{v}{\omega^2}E_z'' - \frac{1}{v}E_z - \frac{\sigma}{2\pi\epsilon_0\omega^2}\delta'(z-a) \Longrightarrow E_z'' - \omega^2\left(\frac{1}{v^2} - \frac{1}{c^2}\right)E_z = -\frac{\sigma}{2\pi v\epsilon_0}\delta'(z-a)
$$

The solutions are of the form

$$
E_z = A_+e^{\kappa z} + A_-e^{-\kappa z}
$$

where

$$
\kappa = \frac{\omega}{v}\sqrt{1 - \frac{v^2}{c^2}} = \frac{\omega}{\gamma v}.
$$

For $z > a$ we keep only the attenuated part, so:

$$
E_z = \begin{cases} Ae^{-\kappa z} & \text{for } z > a \\ Be^{-\kappa z} + Ce^{\kappa z} & \text{for } z < a \end{cases}
$$

The boundary conditions at $z = a$ are

$$
\frac{\sigma}{2\pi v\epsilon_0} = E_z(a^+) - E_z(a^-) = (A-B)e^{-\kappa a} - Ce^{\kappa a},
$$

and

$$
0 = E_z'(a^+) - E_z'(a^-) = -\kappa(A-B)e^{-\kappa a} - \kappa Ce^{\kappa a}
$$

[If there were a jump in E_z' then E_z'' would have contained a term proportional to $\delta(z-a)$, but as we see from the differential equation, it only contains $\delta'(z-a)$.] Combining the equations we get

$$
C = -(A-B)e^{-2\kappa a} \Longrightarrow \frac{\sigma}{2\pi v\epsilon_0} = (A-B)e^{-\kappa a} - Ce^{\kappa a} = -2Ce^{\kappa a}
$$

$$
\Longrightarrow C = -\frac{\sigma}{4\pi v\epsilon_0}e^{-\kappa a}, \quad A - B = \frac{\sigma}{4\pi v\epsilon_0}e^{\kappa a}.
$$

Now we match the boundary conditions at $z = 0$. There, we have

$$
E_z(0^+) = B + C, \qquad E_z'(0^+) = \kappa(C - B),
$$

and

$$E_z(0^-) = D, \qquad E_z'(0^-) = -ikD.$$

The boundary conditions are

$$\epsilon_0 E_z(0^+) = \epsilon E_z(0^-), \qquad \frac{iv}{\omega}E_z'(0^+) = E_x(0^+) = E_x(0^-) = \frac{iv}{\omega}E_z'(0^-).$$

So, E_z has a discontinuity at $z = 0$ but E_z' is continuous. We therefore get

$$\kappa(C - B) = -ikD, \qquad B + C = \frac{\epsilon}{\epsilon_0}D$$

The solution is:

$$2C = \left(\frac{\epsilon}{\epsilon_0} - i\frac{k}{\kappa}\right)D, \qquad D = -\frac{\sigma}{2\pi v\epsilon_0}\left(\frac{\epsilon}{\epsilon_0} - i\frac{k}{\kappa}\right)^{-1} e^{-\kappa a}.$$

The energy loss per unit wire length, per unit traversed length is

$$\begin{aligned}
\frac{dE}{dxdy} &= -\frac{1}{2\pi}\int \vec{E}^* \times \vec{B}\cdot\hat{z}d\omega = -\frac{1}{2\pi}\int_{-\infty}^{\infty} E_x^* B_y d\omega \\
&= -\frac{1}{2\pi}\int (i\frac{v}{\omega}E_z')^*(-v\mu_0\epsilon E_z)d\omega = \frac{1}{2\pi}\int k\frac{v^2\mu_0\epsilon}{\omega}|D|^2 d\omega \\
&= \int \frac{\mu_0\sigma^2 k}{8\pi^3\epsilon_0\omega}\left(\frac{\epsilon^2}{\epsilon_0^2} + \frac{k^2}{\kappa^2}\right)^{-1} e^{-2\kappa a}d\omega
\end{aligned}$$

The integral is over both positive and negative ω's. Converting to only positive frequencies, we get a factor of 2

$$\frac{dE}{dxdy} = \int \frac{\mu_0\sigma^2 k}{4\pi^3\epsilon_0\omega}\left(\frac{\epsilon^2}{\epsilon_0^2} + \frac{k^2}{\kappa^2}\right)^{-1} e^{-2\kappa a}d\omega$$

where

$$\kappa = \frac{\omega}{v}\sqrt{1 - \frac{v^2}{c^2}} = \frac{\omega}{\gamma v}, \qquad k = \omega\sqrt{\mu_0\epsilon - \frac{1}{v^2}}$$

and the integral is over the range of frequencies for which

$$\epsilon(\epsilon) > \frac{1}{\mu_0 v^2}.$$

Note the exponential fall of as a function of a. The power loss is

$$\frac{dP}{dy} = v\frac{dE}{dxdy}.$$