Quantum cohomology of the Hilbert scheme of points in the plane

&

a nonstationary deformation of Jack polynomials

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Hilbert scheme of points in $\mathbb{C}^2$

By definition,

$$\mathrm{Hilb}_n = \{ \text{ ideals } I \subset \mathbb{C}[x,y] \,, \\ \text{such that } \dim_{\mathbb{C}} \mathbb{C}[x,y]/I = n \;\}$$

For example, if

$$S = \{p_1, \dots, p_n\} \subset \mathbb{C}^2$$

are n distinct points, then

$$I(S) = \{f \in \mathbb{C}[x,y], f(p_1) = \cdots = f(p_n) = 0\}$$

is an element of Hilb_n.

First facts about Hilb_n

The Hilbert scheme Hilb_n is a smooth, irreducible, quasiprojective variety of complex dimension $2n$.

An open dense set of Hilb_n is formed by ideals of n distinct points.

The complement ∂ of this open set is a divisor in Hilb_n and

$$D = -\tfrac{1}{2}\partial = c_1(\mathbb{C}[x,y]/I)$$

is the positive generator of $\mathrm{Pic}(\mathrm{Hilb}_n) \cong H^2(\mathrm{Hilb}_n, \mathbb{Z}) \cong \mathbb{Z}$.

Quantum cohomology

Let X be a projective algebraic variety, $\alpha, \beta, \gamma \in H^*(X)$. The quantum product $\star$ in $H^*(X)$ is defined by

$$(\alpha \star \beta, \gamma) = \sum_{d \geq 0} q^d \boxed{\begin{array}{l}\text{(virtual) number of degree } d \text{ rational} \\ \text{curves in } X \text{ meeting } \alpha^\vee, \beta^\vee, \gamma^\vee\end{array}}$$

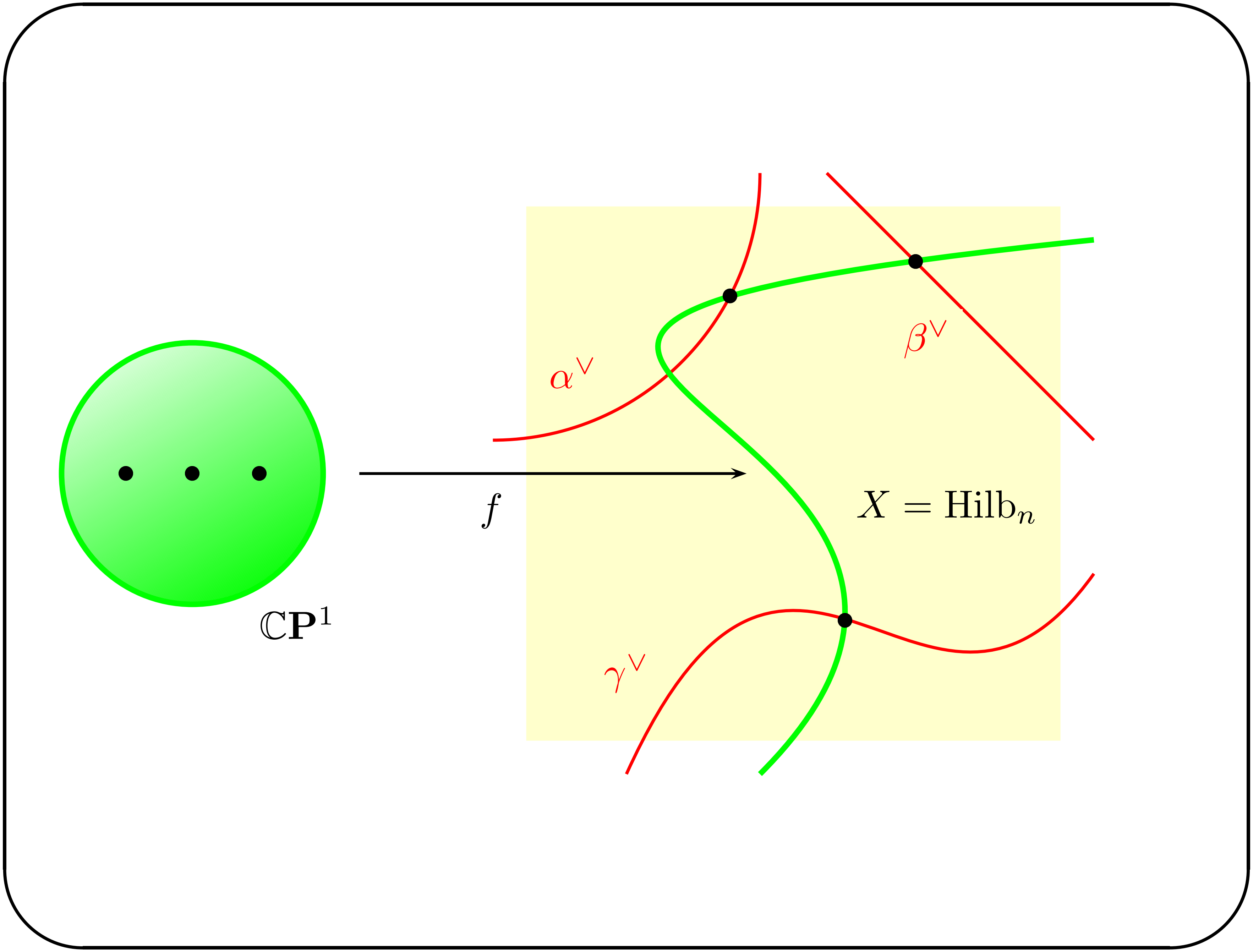
where

$$(\,\cdot\,,\,\cdot\,) = \text{standard inner product on } H^*(X)$$

$$\alpha^\vee = \text{Poincaré dual cycle to } \alpha$$

$$q = \text{new parameter}$$

Setting $q = 0$ gives classical multiplication in $H^*(X)$.



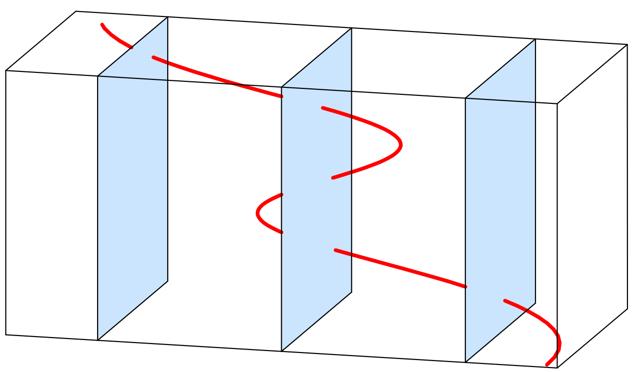
$\mathbb{C}\mathbf{P}^1$
f
$\alpha^\vee$
$\beta^\vee$
$\gamma^\vee$
$X = \mathrm{Hilb}_n$

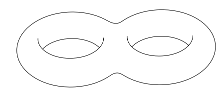
We are interested in quantum cohomology of the Hilbert scheme, in part, because it is related to higher genus curves in 3-folds of the form

$$\text{curve} \times \text{surface}\,, \ \text{e.g. } \mathbf{P}^1 \times \mathbb{C}^2$$

Curves in X = curve $B \times$ surface S

Three points of view:



- parameterized curves, i.e. maps f :  $\to X$
- curves defined by equations, i.e. ideal sheaves on X
- maps from the base B to Hilbert scheme of S

The 1st and 2nd are related by the GW=DT conjecture of [MNOP]. (Now a theorem for rank 2 bundles over curves.)

The 3rd point of view lies between GW and DT.

Review of classical cohomology of Hilb_n

Work of Ellingsrud-Strømme, Nakajima, Grojnowski, Lehn, Vasserot, Li-Qin-Wang, Costello-Grojnowski, . . .

In fact we will need ...

Equivariant cohomology of Hilb_n

The torus $T = (\mathbb{C}^\times)^2$ acts on $\mathbb{C}^2$ and Hilb_n by

$$(z_1, z_2) \cdot (x, y) = (z_1 x, z_2 y)\,.$$

Equivariant cohomology ring $H_T^*(\mathrm{Hilb}_n, \mathbb{Q})$ is a free module over

$$H_T^*(pt, \mathbb{Q}) = \mathbb{Q}[\mathrm{Lie}(T)] = \mathbb{Q}[t_1, t_2]$$

with a basis ...

Nakajima basis

Let $\mu = (\mu_1, \mu_2, \dots)$ be a partition of n with $\ell(\mu)$ nonzero parts.

$$\left|\mu\right\rangle \stackrel{\text{def}}{=} \frac{1}{\prod \mu_i} \overline{\left\{ \boxed{\begin{array}{l}\text{ideals supported at } \ell(\mu) \\ \text{distinct points of } \mathbb{C}^2 \text{ with} \\ \text{multiplicities } \mu_1, \mu_2, \dots\end{array}} \right\}}^{\vee} \in H^{n-\ell(\mu)}$$

For example:

$$\begin{aligned} \left|1^n\right\rangle &= \text{identity} \\ \left|2, 1^{n-2}\right\rangle &= -D \end{aligned}$$

An important role is also played by . . .

Torus-fixed points in Hilb_n

These are ideals I_λ in $\mathbb{C}[x,y]$ spanned by monomials. They correspond to partitions λ of n. Here is $I_{(5,4,2,1)} \in \mathrm{Hilb}_{12}$

1	x	x^2	x^3	x^4	x^5	x^6
y	xy	x^2y	x^3y	x^4y	x^5y	x^6y
y^2	xy^2	x^2y^2	x^3y^2	x^4y^2	x^5y^2	x^6y^2
y^3	xy^3	x^2y^3	x^3y^3	x^4y^3	x^5y^3	x^6y^3
y^4	xy^4	x^2y^4	x^3y^4	x^4y^4	x^5y^4	x^6y^4

In particular, torus-fixed points are isolated.

By Atiyah-Bott, the classes $[I_\lambda]$ of fixed points form a basis of

$$H_T^*(\mathrm{Hilb}_n) \otimes \mathbb{Q}(t_1, t_2) \,.$$

We have

$$[I_\lambda] \cdot [I_\mu] = \text{const}\ \delta_{\lambda\mu}\, [I_\lambda] \,,$$

and, therefore, $[I_\lambda]$ are eigenvectors of multiplication by any element of $H_T^*(\mathrm{Hilb}_n)$.

As it turns out, in the Nakajima basis,

$$\boxed{\text{Classical multiplication by the divisor } D} = \boxed{\text{Quantum Calogero-Sutherland operator}}$$

Quantum Calogero-Sutherland Hamiltonian

$$\mathsf{H}_{CS} = \frac{1}{2}\sum_{i=1}^{N}\left(z_i\frac{\partial}{\partial z_i}\right)^2 + \theta(\theta-1)\sum_{i<j}\frac{1}{|z_i - z_j|^2}$$

describes N identical quantum particles on the circle $|z_i| = 1$ interacting via an inverse square potential.

$$\theta = \text{coupling constant}$$

Ground state: $\psi_0 = \prod_{i<j}(z_i - z_j)^\theta$

$$\widetilde{\mathsf{H}}_{CS} = \psi_0^{-1}\,\mathsf{H}_{CS}\,\psi_0 \quad \text{preserves} \quad \mathbb{C}[z_1,\ldots,z_N]^{S(N)}$$

Convenient to view N as formally infinite.

Identify $\bigoplus_{n\geq 0} H^*(\mathrm{Hilb}_n)$ with symmetric polynomials in z_i by

$$\left|\mu\right\rangle \mapsto \frac{1}{|\operatorname{Aut}(\mu)| \prod \mu_i} p_\mu \,,$$

where

$$p_\mu = \prod p_{\mu_i} \,, \quad p_k = \sum z_i^k \,.$$

Then . . .

Multiplication by D $=$ CS operator

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T-fixed points $[I_\lambda]$ = Jack polynomials J_λ

Multiplication by D	$=$	CS operator
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parameter $-t_1/t_2$	$=$	parameter θ

Multiplication by D	$=$	CS operator
T-fixed points $[I_\lambda]$	$=$	Jack polynomials J_λ
parameter $-t_1/t_2$	$=$	parameter θ
multiplication by H^*	$=$	commuting Hamiltonians

Quantum multiplication by $D = ?$

It is not the Macdonald operator ...

Theorem 1. The operator of quantum multiplication by D is:

diagonal

$$\mathsf{M}_D = \frac{t_1+t_2}{2} \sum_{k>0} k\, \frac{(-q)^k+1}{(-q)^k-1}\, \alpha_{-k}\alpha_k + \frac{1}{2} \sum_{k,l>0} \Big[t_1 t_2\, \alpha_{k+l}\alpha_{-k}\alpha_{-l} - \alpha_{-k-l}\alpha_k\alpha_l \Big] + \frac{t_1+t_2}{2}\, \frac{1-q}{1+q}\, n$$

where

$$\alpha_{-k} = \text{multiplication by } p_k\,, \quad \alpha_k = k\frac{\partial}{\partial p_k}\,.$$

splitting

joining

Main technical issue: while T-fixed points in Hilb_n are isolated, T-invariant curves are not.

A variety of techniques is used to get around this.

Corollary. Divisor D generates the quantum ring over $\mathbb{Q}(t_1, t_2)$

False for classical cohomology

The entire quantum ring structure is thus determined

WDVV then determines all genus 0 GW invariants

Localization determines (in principle, but not very effectively) higher genus GW invariants from genus 0 data

Quantum differential equation, I

$$q\frac{d}{dq}\,\Psi = \mathsf{M}_D\,\Psi$$

Nonstationary analog of Schrödinger equation for Calogero-Sutherland operator.

Solutions are related to:

- general 1-partition triple Hodge integrals in GW theory,
- equivariant vertex with one infinite leg in DT theory

For example, for $n = 2$, QDE is a hypergeometric equation.

For $n = 3$, QDE has the form

$$q\frac{d}{dq}\,\Psi = \begin{bmatrix} 3(t_1+t_2)\dfrac{(q-1)(q+1)^2}{q^3+1} & -3 & 0 \\ 2t_1t_2 & (t_1+t_2)\dfrac{q+1}{q-1} & -1 \\ 0 & 3t_1t_2 & 0 \end{bmatrix}\Psi$$

Quantum differential equation, II

$$q \frac{d}{dq} \Psi = \mathsf{M}_D \Psi$$

Linear ODE with regular singularities at

$$q = 0, \infty, \text{roots of unity}$$

Residues at $q = 0, \infty$ are CS operators.

Residues at $q = \sqrt{1}$ are diagonal in Nakajima basis.

Solubility is controlled by:

$$\text{Monodromy} = ?$$

Theorem 2. Monodromy of QDE is invariant under

$$t_i \mapsto t_i - 1\,,$$

where $i = 1, 2$, provided

$$t_i \neq r/s\,, \quad 0 < r \leq s \leq n\,.$$

Corollary. When the level

$$\kappa = t_1 + t_2$$

is an integer, there is no monodromy around roots of unity.

When $\kappa = 0, 1, 2, \ldots$, solutions of QDE have the form

$$\Psi = q^{-c(\lambda)} \mathsf{Y}^{\lambda}\,,$$

where

$$c(\lambda) = \sum_{\square=(i,j)\in\lambda} \left[(j-1)t_1 + (j-1)t_2\right]$$

are the CS eigenvalues and

$$\mathsf{Y}^{\lambda} = \mathsf{J}_{\lambda} + O(q)$$

is a polynomial in q of degree

$$\deg_q \mathsf{Y}^{\lambda} = c(\lambda) + c(\lambda') \in \mathbb{Z}_{>0} \cdot \kappa$$

First properties of polynomials Y^λ

Symmetry:

$$\mathsf{Y}^\lambda(t_1, t_2) = \mathsf{Y}^\lambda(t_2, t_1)\,.$$

Biorthogonality:

$$\big(\mathsf{Y}^\lambda(t_1, t_2), \mathsf{Y}^\mu(-t_1, -t_2)\big) = \delta_{\lambda\mu} \|\mathsf{J}_\lambda\|^2\,,$$

where

$$(p_\lambda, p_\mu) = \frac{\delta_{\lambda\mu}\, z_\mu}{(t_1 t_2)^{\ell(\mu)}}\,.$$

Note that $\|\mathsf{J}_\lambda\|^2$ is given by the standard hook product.

Scattering

As $q \to 0, \infty$, QDE becomes the CS system.

The $q \to q^{-1}$ symmetry of QDE acts by:

$$\frac{(-q)^{-c(\lambda)}}{h_\lambda} \mathsf{Y}^\lambda(q) = \omega \cdot \frac{(-q)^{c(\lambda')}}{h_{\lambda'}} \mathsf{Y}^{\lambda'}(q^{-1}) \,,$$

where

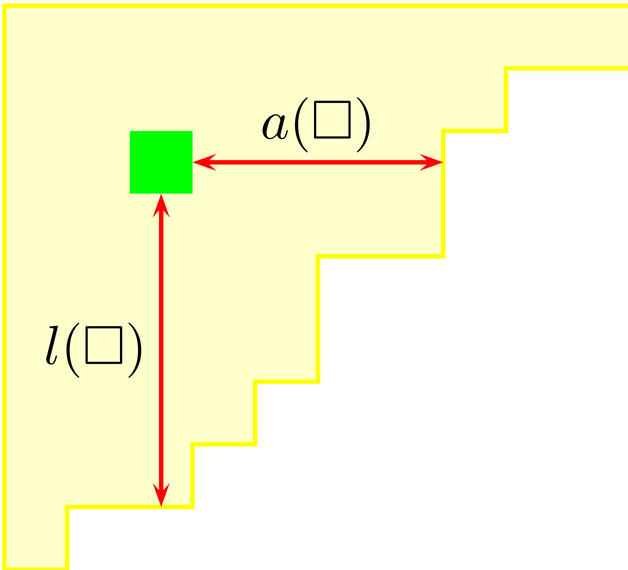
$$\omega \cdot p_k = (-1)^{k-1} p_k$$

is the standard involution on symmetric functions.

The factors h_λ are given by ...

$$h_\lambda = \prod_{\square\in\lambda} \frac{\Gamma\left(t_1(a(\square)+1) - t_2 l(\square) + 1\right)}{\Gamma\left(t_1 a(\square) - t_2(l(\square)+1)\right)}$$

where the arm $a(\square)$ and leg $l(\square)$ of a square $\square$ are defined by



Note that there are $\kappa + 1$ factors per square in h_λ.

Formula for $\mathsf{Y}^\lambda = \,?$

At $q = 0$, the coefficients of p_μ in J_λ (Jack analogs of symmetric group characters χ^λ_μ) have not been understood.

However, we expect that at $q = -1$ the coefficients are nice.

Namely, they are χ^λ_μ times an explicit product of linear forms in t_i.

All this indicates that there is probably a rich theory of the polynomials Y^λ. Maybe this is the beginning of the theory of

nonstationary integrable systems ?

We expect more interesting examples from

GW/DT theory of threefolds.