Topological Strings and Black Holes

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based on:

A. Strominger, C. Vafa + H.O. hep-th/0405146

M. Aganagic, N. Saulina, C. Vafa + H.O. hep-th/0411280

C. Vafa, E. Verlinde + H.O. hep-th/0502211

R. Dijkgraaf, R. Gopakumar, C. Vafa + H.O. hep-th/0504221

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Topological String Theory

(1) Start with a Calabi-Yau 3-fold.

$\mathcal{N}=2$ superconformal sigma-model

$$\Sigma_g \to CY_3$$

(2) Topologically twist the sigma-model.

(3) Couple it to the topological gravity on Σ_g .

$$F_g = \int_{\mathcal{M}_g} \langle \, | \prod_{i=1}^{3g-3} (\eta_i, G^-) |^2 \rangle$$

supercurrent

Beltrami differential

$$Z^{top}_{pert.} = \exp\Big[\sum_{g=0}^{\infty} \lambda^{2g-2} F_g \Big]$$

The series is typically not summable.

Topological String Partition Function = Wave Function

Consider the topological B-model

moduli space $\mathcal{M}_B$: z^i complex structure

λ topological string coupling

tangent space to $\mathcal{M}_B$ = $H^3(CY_3, \mathbb{R})$

$$\dim_{\mathbb{C}} H^3 = \underbrace{h^{2,1}}_{\delta z^i} + \underbrace{1}_{\lambda}$$

For a given background, we can define the B-model.

F_g is also a holomorphic function of $H^3(CY_3)$.

$$F_g \sim \left\langle \int_{\mathcal{M}_g} (G_i^-)^{3g-3} (G_{\bar{\imath}}^-)^{3g-3} \, e^{x^I \int_{\Sigma_g} \mathcal{O}_I} \right\rangle$$

$x^I \in H^3$, $I = 0, 1, \cdots, h^{2,1}$

$$\psi_{top}(x^I; z^i, \bar{z}^i) = \exp\left(\sum_g F_g\right)$$

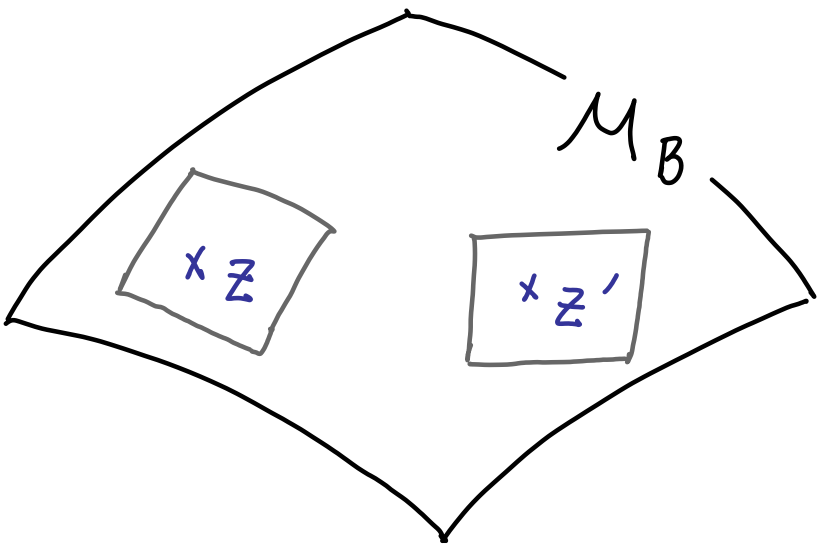
x^I including λ

The holomorphic anomaly equations

derived in BCOV,
interpreted by Witten

$$\frac{\partial}{\partial z^I} \psi_{top}(x; z, \bar{z}) = \left(\frac{\partial}{\partial x^I} + \cdots \right) \psi_{top}$$

$$\frac{\partial}{\partial \bar{z}^I} \psi_{top}(x; z, \bar{z}) = \left(\bar{C}_I{}^{JK} \frac{\partial^2}{\partial x^J \partial x^K} + \cdots \right) \psi_{top}$$



Interptetation:

(1) On each tangent space, there is a Hilbert space.

(2) The holomorphic anomaly equation is describing the paralell transport between tangent spaces at different points.

More on (1):

- $H^3(CY_3, \mathbb{R})$ has a symplectic structure.
- Topological string uses a holomorphic polarization.

$$\underbrace{H^{3,0} \oplus H^{2,1}}_{x^I} \oplus H^{2,1} \oplus H^{0,3}$$

More on (2):

- The polarization depends on $(z^i, \bar{z}^i)$

$\Rightarrow$ Wave-functions are related by Fourier transformation.

topological string partition function $Z_{top} = \exp\left(\sum_g F_g\right)$

= wave function $\psi_{top}(x ; z, \bar{z})$

for quantization of $H^3(CY_3, \mathbb{R})$

(H^3 = tangent space to the moduli space of CY3)

On the other hand,
the topological string partition function gives superpotential terms for CY3 compactification of type II superstring. *(BCOV)*

Can we intepret the topological string partition function as a physical wave function in type II superstring?

Black Hole Entropy

Consider type IIB superstring on $CY_3 \times \mathbb{R}^{3,1}$

RR 4-form $\Rightarrow$ gauge field A_μ^I in 4d.

$I = 0, 1, \dots, h^{2,1}$

D branes wrapping on 3 cycles in CY_3

q_I times on A_I

p^I times on B^I

$A_I, B^I \in H_3(CY_3)$

$A_I \cap A_J = B^I \cap B^J = 0$

$A_I \cap B^J = \delta_I{}^J$

= BPS black hole in four dimensions

with electric charges q_I, magnetic charges p^I

Conjecture (Strominger, Vafa + H.O.)

$$Z_{BH} \equiv \sum_q \Omega(p, q)\, e^{-q\phi}$$

$$= |\psi_{top}(X)|^2$$

where $X^I = p^I + \frac{i}{\pi}\phi^I$

Black hole partition function:

$$Z_{BH} = \sum_q \Omega(p,q)\, e^{-q\cdot\phi}$$

p : magnetic charges of the black hole

ϕ : chemical potentials for electric charges

Perturbative topological string partition function:

$$\psi_{top}^{(pert.)} = \exp\left[\sum_g F_g(X) \right]$$

$\lambda = 4\pi i / X^0$: string coupling constant

X^i / X^0 : complex structure of CY_3

Black Hole Charges <==> Calabi-Yau Moduli.

$$X^I = p^I + \frac{i}{\pi}\phi^I \qquad (I = 0, 1, \cdots, h^{2,1})$$

$$Z_{BH}(p,\phi) = |\psi_{top}(X)|^2$$

Why?

The perturbative topological string amplitudes give low energy effective action terms.

BCOV/1994

It turns out that these are the terms that are relevant in computing perturbative string corrections to the Bekenstein-Hawking entropy formula.

Lopez-Cardoso, de Wit + Mohaupt/1998-99

Define $$\mathcal{F} = \sum_g F_g(X) + \sum_g \bar{F}_g(\bar{X})$$

$(X^I : \mathcal{N}=2 \text{ chiral superfield})$

Black Hole Attractor Eguations:

$$X^I = p^I + \frac{i}{\pi}\phi^I, \qquad q_I = \frac{\partial}{\partial\phi^I}\mathcal{F}(p,\phi)$$

Black Hole Entropy (all order in string perturbation):

$$S_{BH}(p,\phi) = -q_I\phi^I + \mathcal{F}(p,\phi)$$

This is the Legendre transformation:

$$q \longleftrightarrow \phi$$

Entropy and Wave Function

The OSV conjecture can be inverted as:

$$\Omega(p, q) = \langle \psi_{top} | e^{-q\phi - p\tilde{\phi}} | \psi_{top} \rangle$$

where

$$\langle \psi_1 | \psi_2 \rangle = \int d\phi \, \overline{\psi}_1(\phi) \, \psi_2(\phi)$$

$$\tilde{\phi} = -i \frac{\partial}{\partial \phi}$$

This is the Wigner transform

wave-function $\psi(\phi)$

$$\Downarrow$$

function $\Omega(p, q)$ on the phase space

The OSV conjecture as AdS/CFT correspondence

(1) AdS story:

The near horizon geometry of the black hole is

$$AdS_2 \times S^2 \times CY_3$$

$$|\psi_{top}|^2 =$$ Partition function of type II string in this geometry

(2) CFT story:

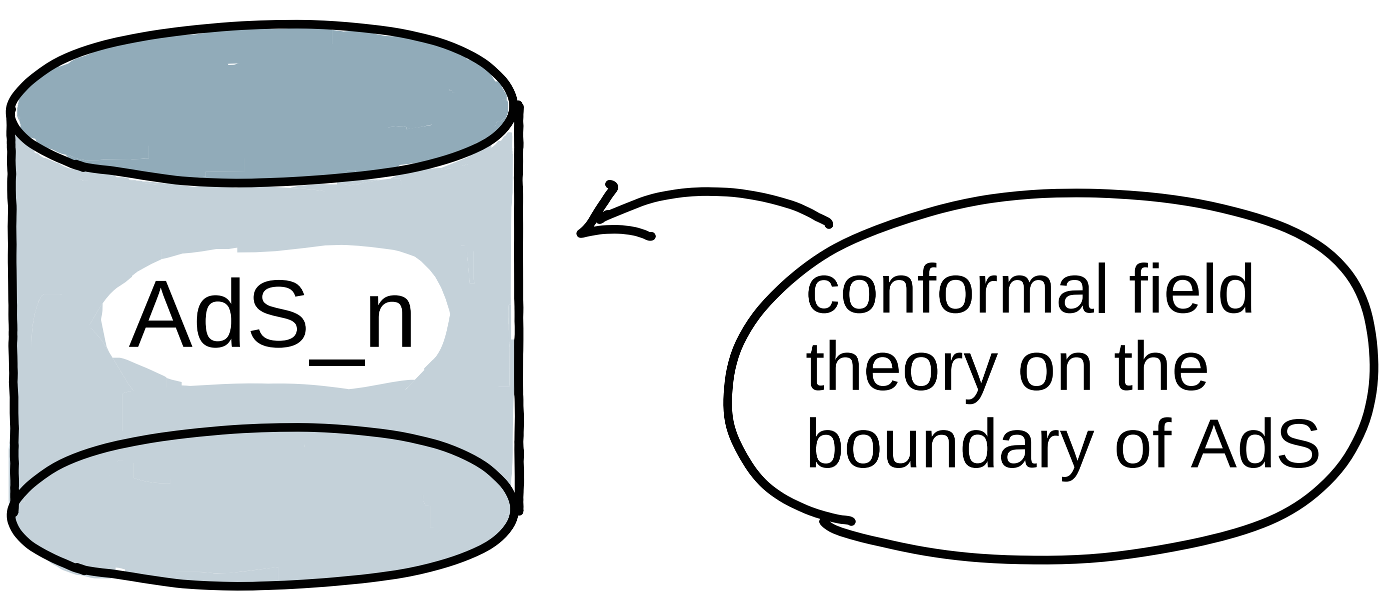
$$\Omega(p, q) =$$ Number of BPS states on the D brane worldvolume

Gauge theory computation on the D branes.

AdS_n / CFT_{n-1}

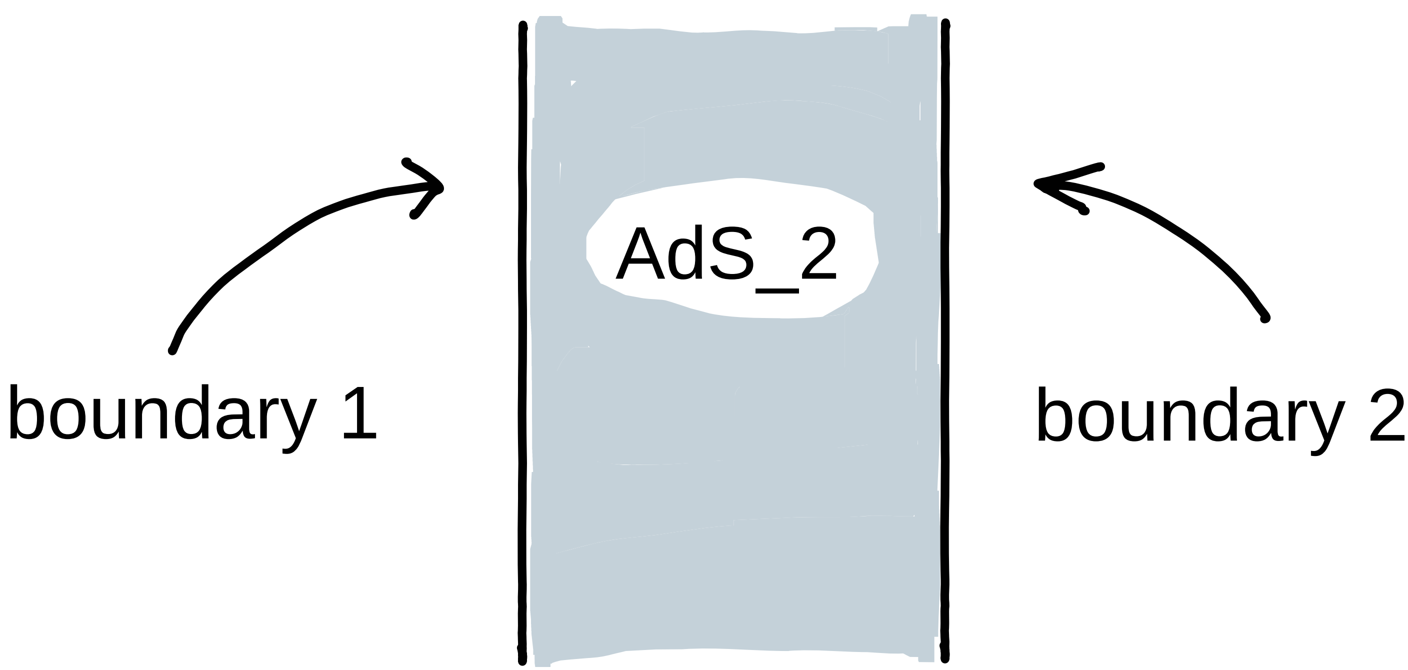
Superstring theory on
n-dim anti de Sitter space
times a compact space

is equvalent to
a conformal field theory
in (n-1) dimensions.



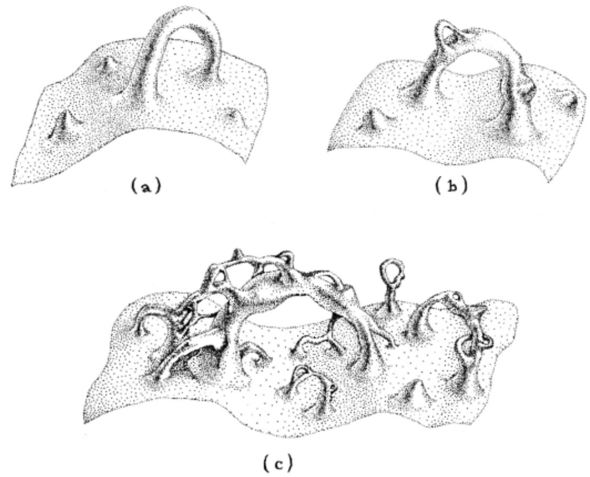
In our case, we have
2-dimensional AdS.

AdS_2 has two boundaries.

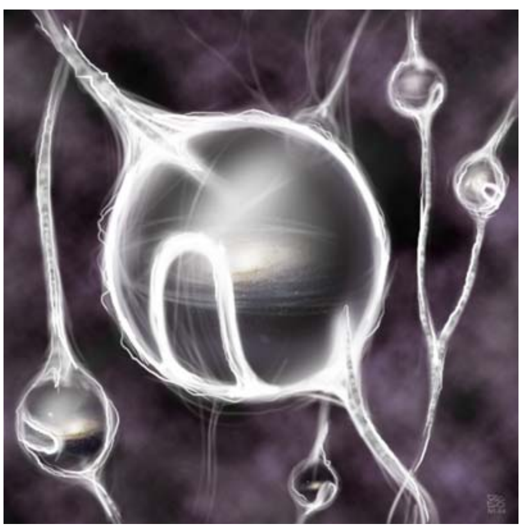


Questions:

Does the quantum gravity involves a sum over topologies of spacetime?



How about a sum over disconnected components of spacetime?



--- baby universes?

Can we maintain the unitarity and the quantum coherence?

An example:

This example is based on the A-model,
so we will take the mirror of the story.

Consider D branes in type IIA string theory
wrapping 0, 2, 4, and 6 cycles of a Calabi-Yau 3-fold.

q_0 = # 0 branes

q_i = # 2 branes $\qquad i = 1, 2, \cdots, h^{1,1}$

p^i = # 4 branes

p^0 = # 6 branes

$\lambda = \frac{4\pi i}{X^0}$: topological string coupling

$t^i = \frac{X^i}{X^0}$: Kähler moduli

$$CY_3 : \mathcal{O}(-p) \oplus \mathcal{O}(p+2g-2) \to \Sigma_g$$

Two line bundles of degrees *-p* and *p+2g-2* over a genus *g* Riemann surface

The total space is a Calabi-Yau manifold.

$Z^{top}_{pert.}$ Topological string amplitudes on this CY was recently computed to all order in the perturbative expansion.

Bryan + Pandharipande/2004

Z_{BH} Consider *N* D4 branes on the total space of the degree *-p* bundle over the Riemann surface.

$$\mathcal{O}(-p) \to \Sigma_g$$

$$X^0 = \frac{4\pi i}{\lambda} \quad (\text{no } D_6 \text{ charge})$$

$$X^1 = \underbrace{(p+2g-2)N}_{D_4 \text{ charge}} + \frac{i}{\pi}\phi^1$$

We want to compute the partition function of the N=4 super Yang-Mills on this 4-manifold.

When g=1, $\mathcal{O}(-p) \to \Sigma_{g=1}$

the relevant gauge theory is the *U(N)* Yang-Mills theory on the two-dimensional torus.

This is equivalent to *N* non-relativistic free fermions on a circle

$$H = \sum_{i=1}^{N} \frac{1}{2} p_i^2$$

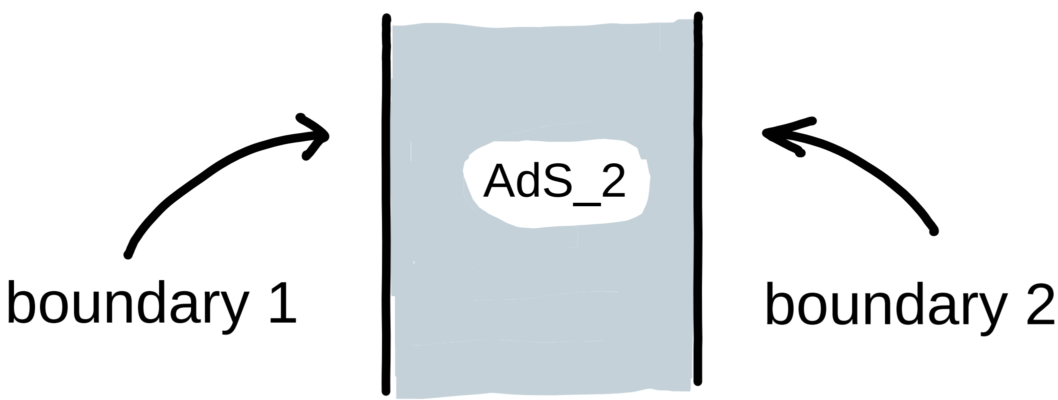
$$p_i = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$$

The dual gravity theory
is type II superstring theory on

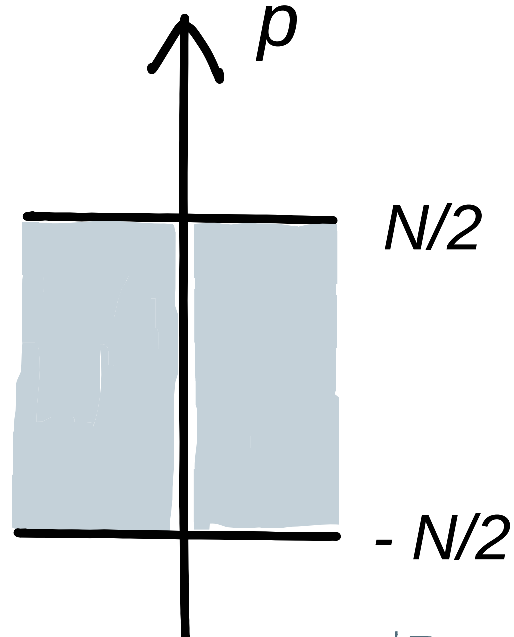
AdS_2 x S_2 x CY_3

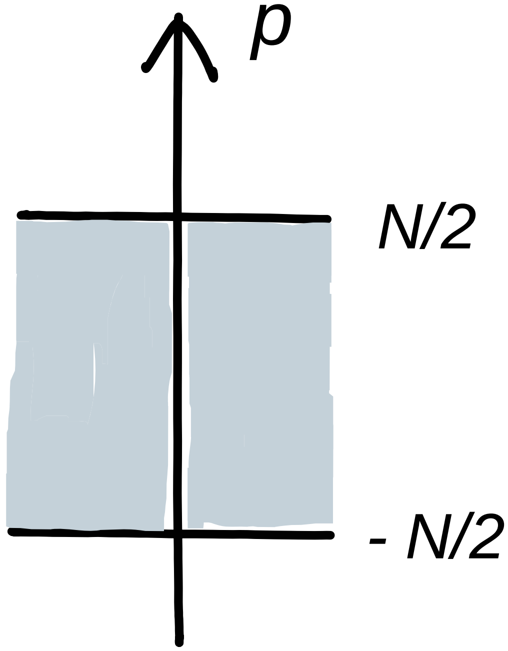
2d anti de Sitter 2d sphere 6d Calabi-Yau

AdS_2 has two boundaries.



This is dual to the fact that the *N* fermion theory has two fermi surfaces.





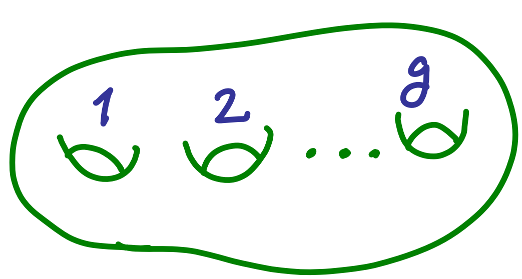
When N is large, the two fermi surfaces decouple from each other.

In the large N limit, fluctuations of each fermi surface are described by free relativistic fermions.

The $1/N$ expansion of the non-relativistic free fermion partition function correctly reproduces the string perturbation theory

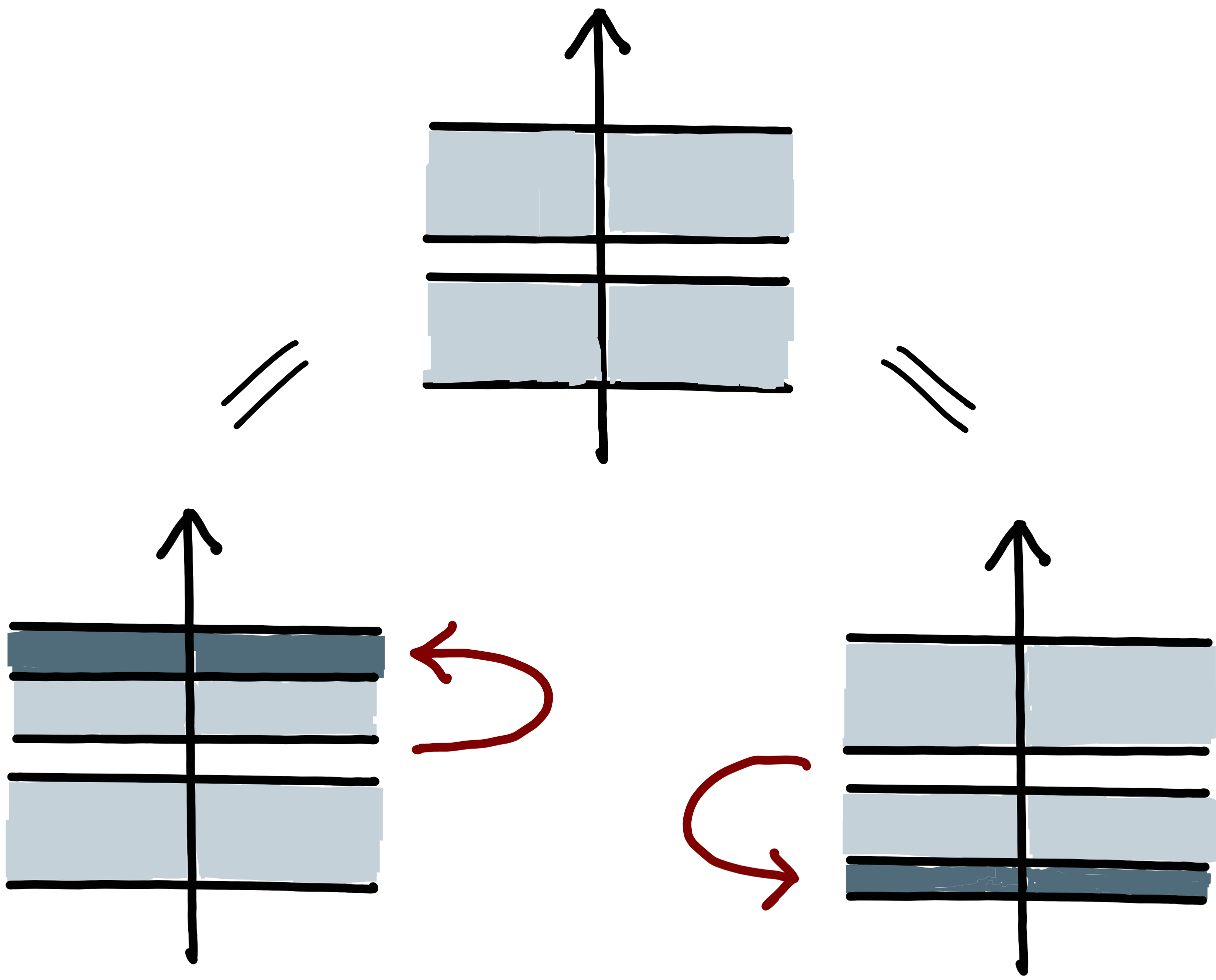
in AdS_2 x S_2 x CY_3

$$\exp\left(\sum_{g=0}^{\infty} \frac{1}{N^{2g-2}} \right.$$



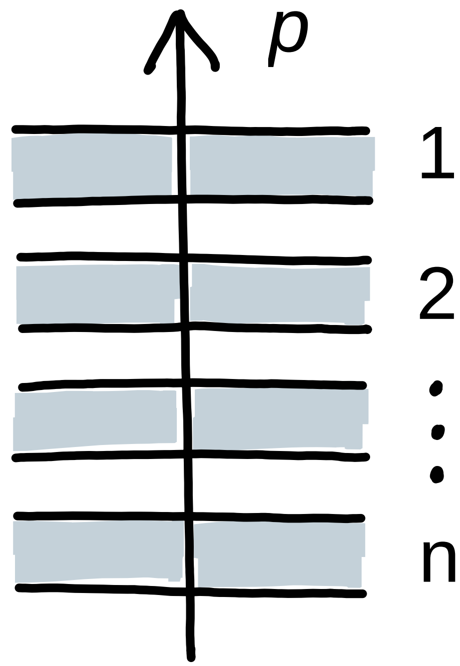
$$\left. \vphantom{\sum_{g=0}^{\infty}} \right)$$

For finite N,
the two fermi surfaces are entangled by excitations that are non-perturbative in $1/N$.

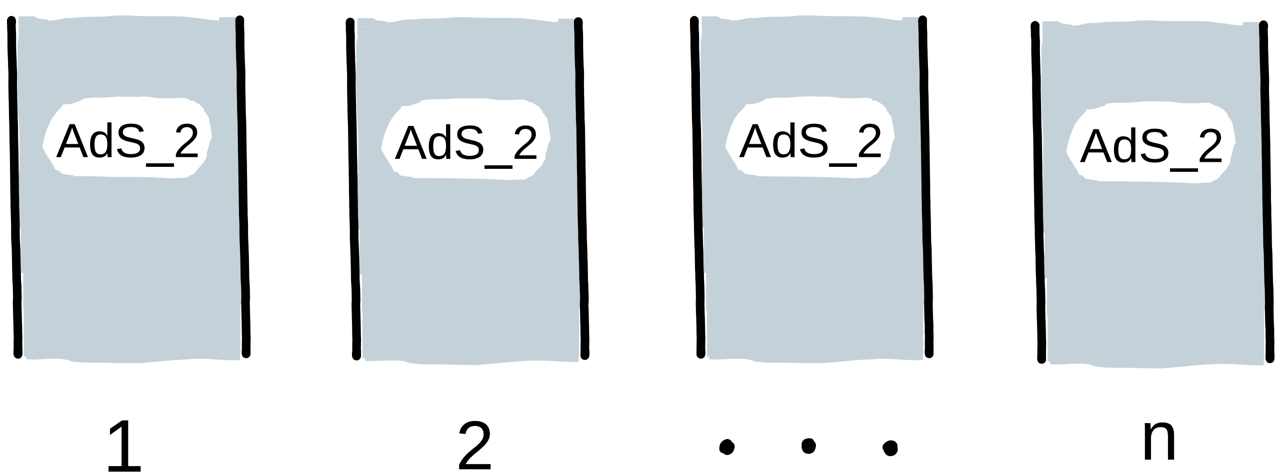


These non-perturbative states would be over-counted if we ignore the entanglement of the two fermi surfaces.

Configurations with 2n fermi surfaces



is dual to n disjoint universes



weighted by the Catalan number of planar binary trees with n branches (remembering how baby universe are created from the parent universe).

Lessons for Quantum Gravity

In this example, the *1/N* expansion of *N* non-relativistic free fermions correctly reproduces the perturbative string theory.

In the fermion theory, $O(e^{-N})$ effects entangle two fermi surfaces.

In the gravity theory, they correspond to creation of baby universes.

Unitarity of quantum gravity can be maintained after we sum over topologies (including sum over disjoint universes).

This would be relevant for the black hole information paradox.

Baby universes do not destroy quantum coherence, in accord with a general argument by Coleman.

Fin