$\mathcal{H}$ - space of observables

$e_1, \dots, e_n$ - basis of $\mathcal{H}$

$\eta_{\alpha\beta} = \langle e_\alpha e_\beta \rangle$

$c_{\alpha\beta\gamma} = \langle e_\alpha e_\beta e_\gamma \rangle$

$c^\alpha_{\beta\gamma}(\tau) = \eta^{\alpha\lambda} c_{\lambda\beta\gamma}(\tau)$ -

- structure constants of commutative associative algebra

In flat coordinates $\eta_{\alpha\beta}$ does not depend on τ and

$$c_{\alpha\beta\gamma}(\tau) = \partial_\alpha \partial_\beta \partial_\gamma F(\tau)$$

$$\frac{\partial^3 F(\tau)}{\partial\tau^\alpha \partial\tau^\beta \partial\tau^\delta} \eta^{\delta\gamma} \frac{\partial^3 F(\tau)}{\partial\tau^\gamma \partial\tau^\omega \partial\tau^\rho} = \frac{\partial^3 F(\tau)}{\partial\tau^\alpha \partial\tau^\omega \partial\tau^\delta} \eta^{\delta\gamma} \frac{\partial^3 F(\tau)}{\partial\tau^\gamma \partial\tau^\beta \partial\tau^\rho}$$

WDVV eqns

$\eta_{\alpha\beta} = c_{1\alpha\beta}$, e_1 is a unit element

$\tau_1, \dots, \tau_n$ - coordinates on $\mathcal{T}$

$\mathcal{T}$ - Frobenius manifold

$\nabla_\alpha = \partial_\alpha - z^{-1}\hat{C}_\alpha$

$\hat{C}_\alpha$ - operator with matrix $C_{\alpha\gamma}^{\beta}$

$[\nabla_\alpha(z), \nabla_\beta(z)] = 0$

$$\underline{(\nabla_\alpha(z)\varphi, \psi) + (\varphi, \nabla_\alpha(-z)\psi) = \partial_\alpha(\varphi, \psi)}$$

$$\underline{\nabla_\alpha(z)S(\tau, z) = 0}$$

$$\partial_\alpha S_\gamma^\beta = z^{-1} C_{\alpha\gamma}^{\varepsilon} S_\varepsilon^\beta$$

$$\begin{cases} S(z=\infty) = 1 \\ S(\tau, z)S^*(\tau, -z) = 1 \end{cases}$$

$\tilde{S}(\tau, z) = S(\tau, z)M(z)$ - another solution

$$M(\infty) = 1$$

$$\underline{M(z)M^*(-z) = 1}$$

$x^\alpha(\tau, \infty) = \tau$
calibration

$$S_\beta^\alpha(\tau, z) = \partial_\beta x^\alpha(\tau, z)$$

$$\partial_\beta \partial_\gamma x^\alpha = z^{-1} C_{\beta\gamma}^{\varepsilon} \partial_\varepsilon x^\alpha$$

$\mathcal{B}$ - twisted loop group =
group of $GL(\mathcal{H})$-valued holomorphic functions on $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ obeying

$$B(z)B^*(-z) = 1$$

$\mathcal{A}$ - group of $Aff(\mathcal{H})$-valued functions on $\mathbb{C}^\times$ with linear part in $\mathcal{B}$.

$\mathfrak{B}$ and $\mathfrak{A}$ - corresponding Lie algebras

$(F(\tau), x^\alpha(\tau, z))$ - calibrated solution of WDVV

$$\boxed{\begin{aligned} \delta F &= \frac{i}{2\pi} \int_\Gamma \left[\frac{b_{\alpha\beta}(\zeta) x^\alpha(\tau,\zeta)}{2} + d_\beta(\zeta) \right] x^\beta(\tau,-\zeta)\, d\zeta \\ \delta x^\alpha &= \frac{1}{2\pi i} \int_\Gamma \frac{\eta^{\delta\varepsilon}[b_{\lambda\delta}(\zeta) x^\lambda(\tau,\zeta) + d_\delta(\zeta)]}{\zeta - z} \cdot \\ &\quad \cdot \partial_\varepsilon x^\sigma(\tau,-\zeta)\, \partial_\sigma x^\alpha(\tau,z)\, d\zeta \end{aligned}}$$

- Γ - circle with center at $z=0$ with radius $< |z|$

$\boxed{(F+\delta F, x+\delta x) \text{ is a calibrated solution of WDVV}}$
if $(b,d) \in \mathfrak{A}$

If we require that e_1 is the unit element, $\eta_{\alpha\beta} = c_{1\alpha\beta}$, then we should assume that $d_\alpha(z) = z\, b_{1\alpha}(z)$.

<u>Lie algebra $\mathcal{B}$ acts on the space of Frobenius manifolds equipped with calibration</u>

(Givental, in semisimple case van de Leur)

The same ~~gro~~ Lie algebra acts on the space of TQFT coupled to gravity (of topological string theories) (Givental, Kontsevich – talk in Miami).

It should act also on the space of open-closed topological string theories

Geometry of Frobenius manifolds

$x^\alpha(\tau, z)$ are defined up to affine transformations.

Family of affine structures T_z on manifold J depending on $z \in \mathbb{P}^1 - \{0\}$

$\mathbb{P}^1 \times J$ - holomorphic bundle over $\mathbb{P}^1$; all fibers except the fiber over $z=0$ are affine spaces.

Affine structure $\Leftarrow$ torsion-free flat connection $\Gamma^\gamma_{\alpha\beta}$

Consider holomorphically trivial bundle over $\mathbb{P}^1$ where all fibers except the fiber over $z=0$ are equipped with affine structure and the affine structure is define by means of torsion-free flat connection $\Gamma^\gamma_{\alpha\beta}(z)$ having a pole of order 1 at $z=0$;

Then $\Gamma^\gamma_{\alpha\beta}(z) = z^{-1} C^\gamma_{\alpha\beta}$ after trivialization

The metric $\eta_{\alpha\beta}$ on $\mathcal{T}$ induces a pairing between tangent spaces of T_z and T_{-z} for $z \neq 0$. (Notice that the tangent spaces at all points of affine space are identified.) This pairing tends to the pairing in the tangent space to T_0 as points of T_z and T_{-z} tend to a point of T_0 staying on a holomorphic section of the bundle over a neighborhood of $z = 0$.

Conversely, assume that in a holomorphically trivial bundle over $\mathbb{P}^1$ we have a family of connections on fibers T_z over $z \in \mathbb{P}^1 - \{0\}$ having first order pole at $z = 0$. these connections are flat and torsion-free. Suppose that we have a pairing between tangent spaces of T_z and T_{-z} with appropriate behavior as $z \to 0$. Then one can construct a solution to WDVV.

Symmetries of WDVV

Construct a new holomorphic bundle over $\mathbb{P}^1$ twisting the direct product $\mathbb{P}^1 \times T$ over

$$\mathbb{C}^* = \underset{|z|<\infty}{\mathcal{D}_0} \cap \underset{|z|>0}{\mathcal{D}_\infty}.$$

Total space of the new bundle is obtained by means of the identification $(z, \tau) \sim (z, f_z(\tau))$. We assume that $f_z : T_z \to T_z$ is an affine transformation: $f_z = (B(z), d(z))$. To guarante the existence of pairing between tangent spaces to T_z and T_{-z} we require $B(z)\, B^*(-z) = 1$.

Calculation.
To trivialize the new bundle we find holomorphic sections

$$(\delta^\mu_\lambda + b^\mu_\lambda(z))x^\lambda(\kappa(z), z) + d^\mu(z) =$$

$$= x^\mu(\alpha(z), z)$$

$\alpha^s(z) = \tau^s + a^s(z)$, inner disk D_0

$\kappa^s(z) = \tau^s + k^s(z)$, outer disk D_∞

$k(\tau, \infty) = 0$

We can express a^s and k^s in terms of Cauchy integral

$$\frac{1}{2\pi i} \int_\Gamma \frac{\varphi^s(\zeta)}{\zeta - z} d\zeta$$

where

$$\varphi^s(\zeta) = a^s(\zeta) - k^s(\zeta) =$$

$$= (S^{-1}(\zeta))^s_\mu \left[b^\mu_\lambda(\zeta)x^\lambda(\zeta) + d^\mu(\zeta)\right]$$