05/04

Towards a proof of the Razumov–Stroganov conjecture

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- The $O(1)$ loop model ($TL_{\beta=1}$)
- ASM, 6v, FPL:

◇ Alternating Sign Matrices

◇ 6-vertex model with Domain Wall Boundary Conditions; Izergin–Korepin/Okada formulae

◇ Fully Packed Loop configurations. Link patterns.

- Razumov–Stroganov conjecture
- Multi-parameter generalization and proof of sum rule

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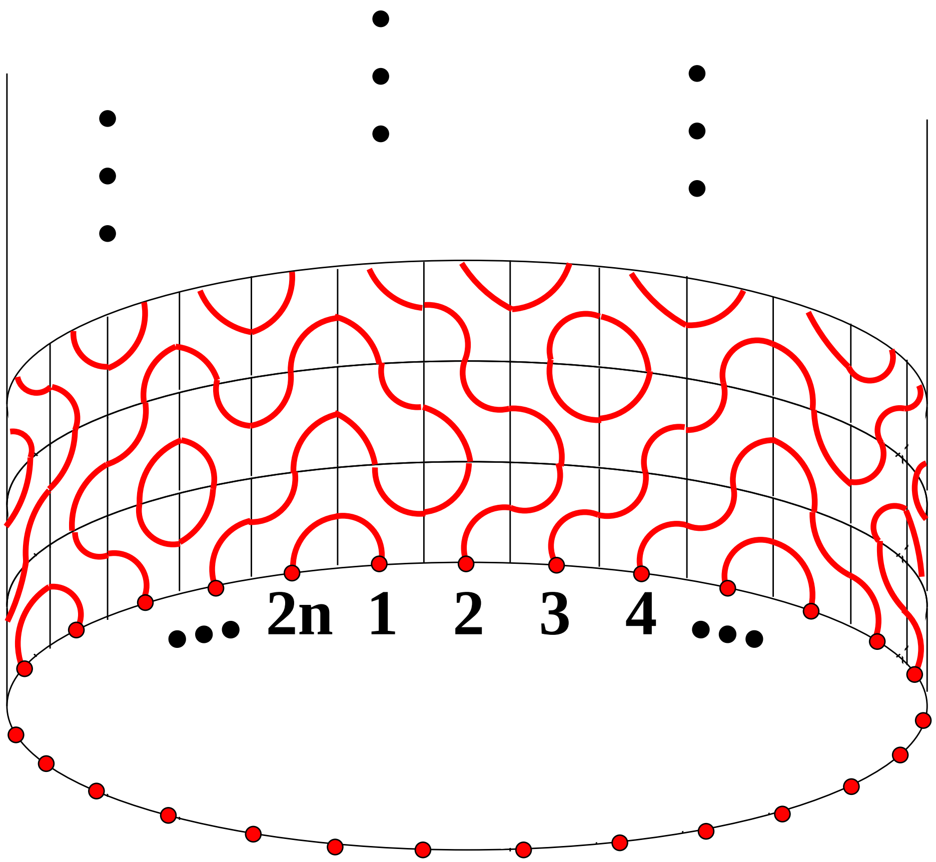
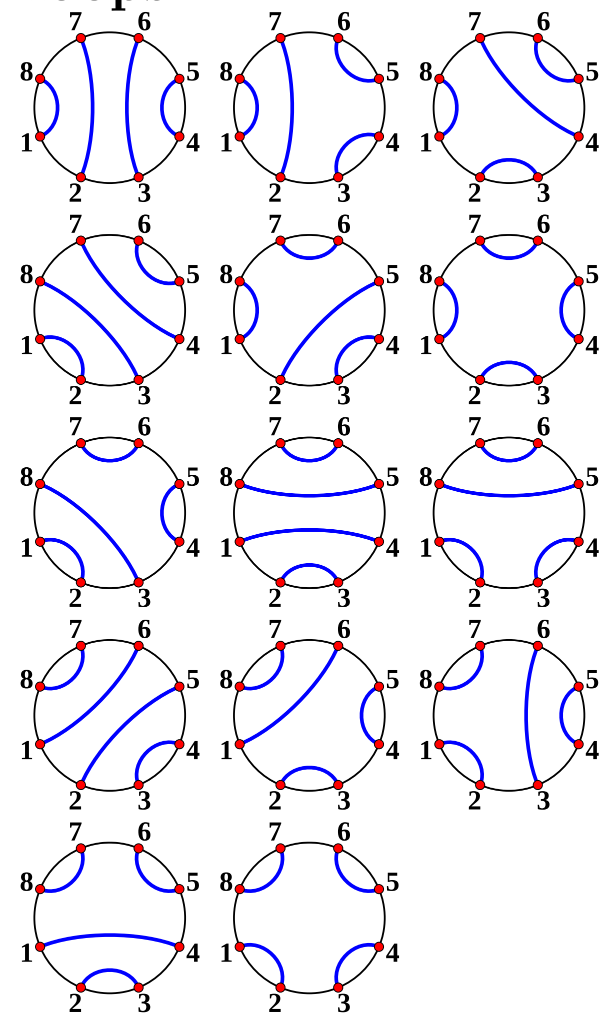
Temperley–Lieb model of loops





What is the probability that external vertex i is connected to vertex j? (plaquettes: t, $1-t$)

$\rightarrow$ Introduce a vector $|\Psi_n\rangle$ whose components are indexed by **link patterns**.

Temperley–Lieb model of loops cont'd

Eigenvector equation:

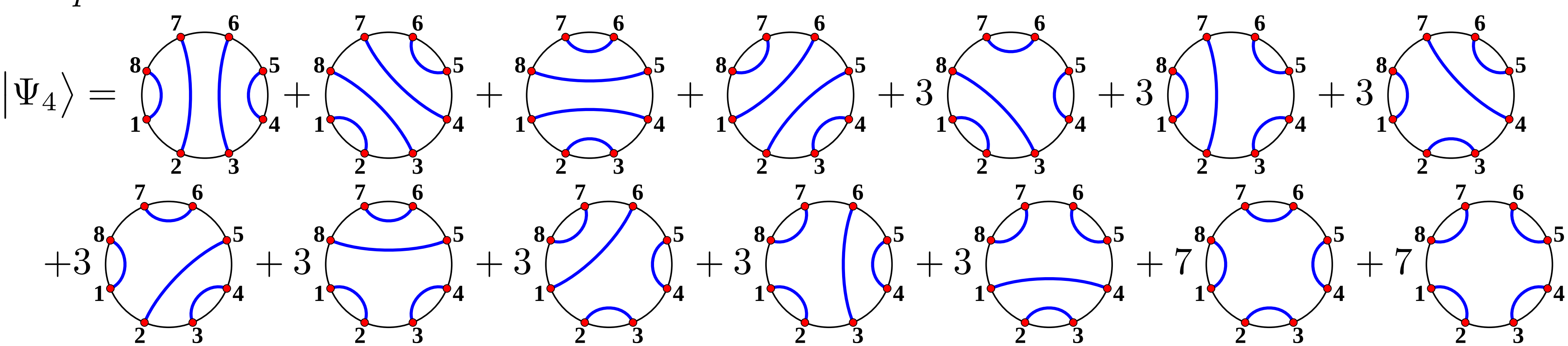
$$T_n(t)\,\big|\Psi_n\big\rangle = \big|\Psi_n\big\rangle$$

where $T_n(t)$ is the **transfer matrix** which adds a row to the semi-infinite cylinder.

One can show that $[T_n(t), T_n(t')] = 0$ for all t, t'.

Note that one can normalize $\big|\Psi_n\big\rangle$ so that its entries are *positive integers* (the actual probabilities are obtained by dividing by the sum of entries).

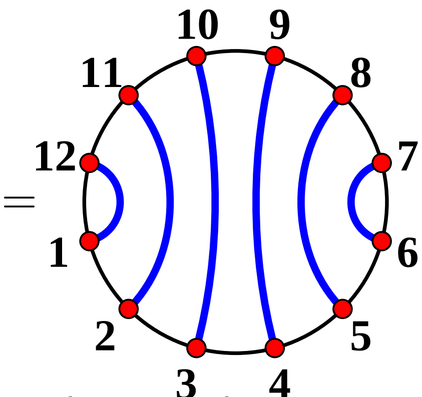
Example: $n = 4$



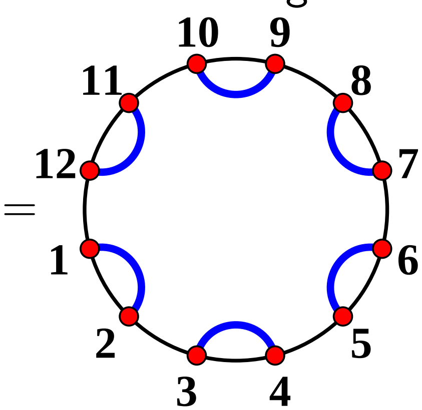
These components are the subject of various conjectures . . .

Some partial conjectures

Batchelor, de Gier, Nienhuis ('01):

(1) The smallest components correspond to patterns $\pi =$ 

and can be set to 1 in such a way that all other components are integer.

(2) The largest components correspond to patterns $\pi =$ 

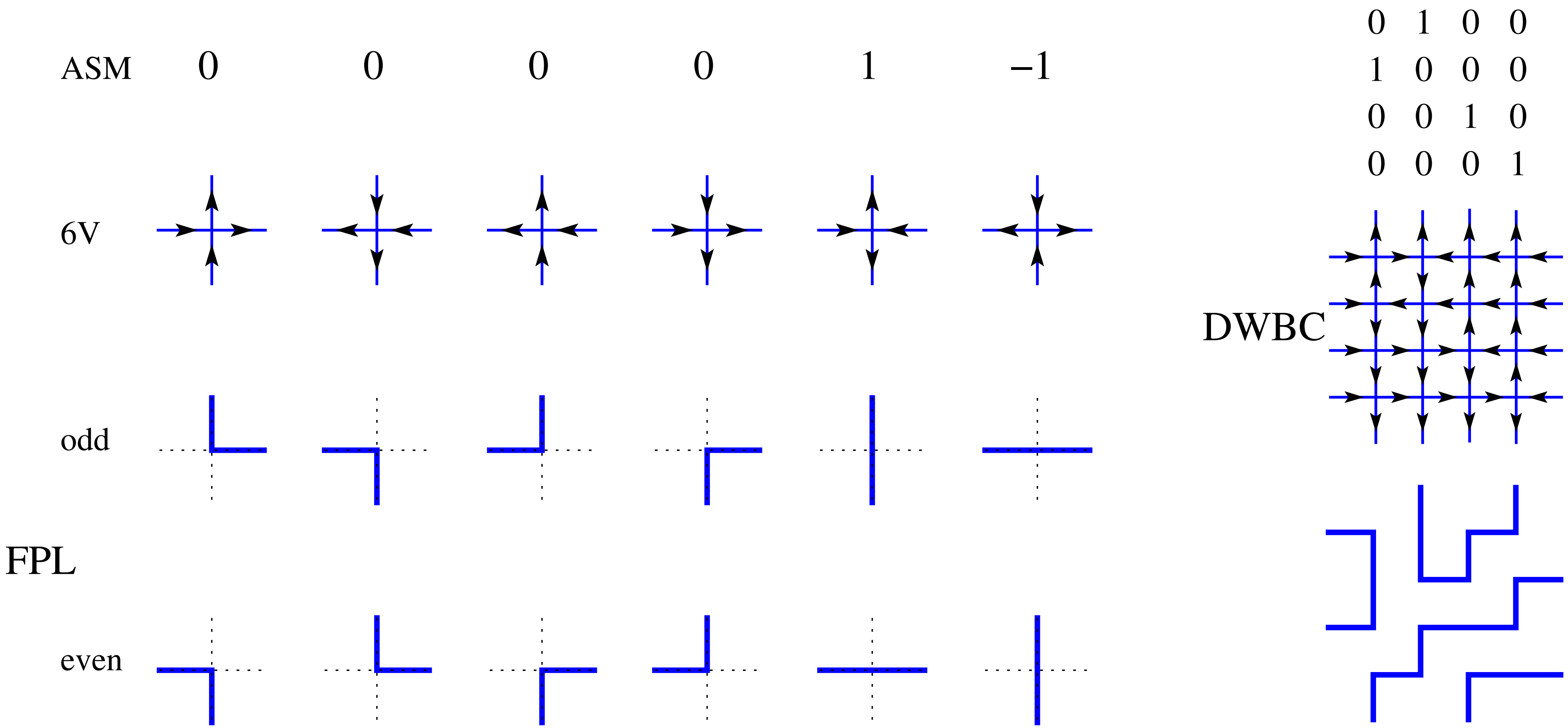
and are equal to A_{n-1}.

(3) The sum of all components $\sum_{\pi} \Psi_n(\pi) = A_n$.

In (2) and (3), A_n is the number of $n \times n$ **Alternating Sign Matrices**: (Zeilberger, '98)

$$A_n = \frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!(n+2)!\cdots(2n-1)!}$$

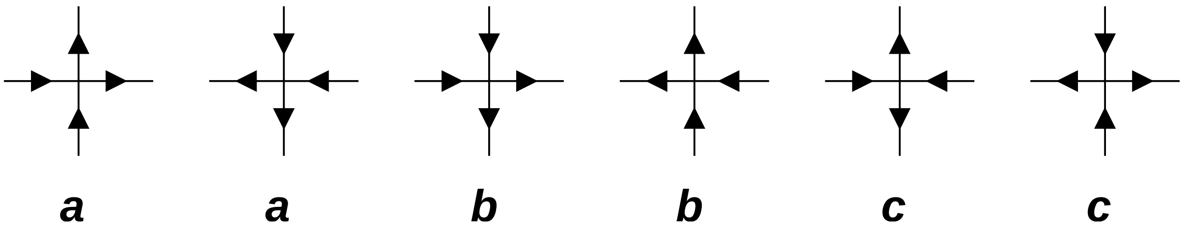
ASM ↔ 6 Vertex ↔ FPL



6 Vertex Model with DWBC: Izergin–Korepin formula

Associate to each horizontal line of the grid a parameter x_i and to each vertical line a parameter y_i.

The weight $w(x, y)$ at a vertex depends on the parameters x, y of the lines and is equal to:



$$a(x, y) = q^{-1/2}x - q^{1/2}y \qquad b(x, y) = q^{-1/2}y - q^{1/2}x \qquad c(x, y) = (q^{-1} - q)(x\,y)^{1/2}$$

$$A_n(x_1, \ldots, x_n; y_1, \ldots, y_n) \equiv \sum_{\text{6v DWBC configs}} \prod_{i,j=1}^{n} w(x_i, y_j)$$

Izergin–Korepin formula ('87):

$$A_n(x_1, \ldots, x_n; y_1, \ldots, y_n) = \frac{\prod_{i,j=1}^{n} a(x_i, y_j) b(x_i, y_j)}{\prod_{i<j}(x_i - x_j)(y_i - y_j)} \det_{i,j=1\ldots n} \left(\frac{c(x_i, y_j)}{a(x_i, y_j) b(x_i, y_j)} \right)$$

NB: $A_n(x_1, \ldots, x_n; y_1, \ldots, y_n)$ is a symmetric function of the x_i, and of the y_i.

Kuperberg ('98): set $q = \mathrm{e}^{2i\pi/3}$ and $x_i = y_i = 1 \Rightarrow$ recover Zeilberger's formula for A_n.

6 Vertex Model with DWBC at $q = \mathrm{e}^{2i\pi/3}$: Okada formula

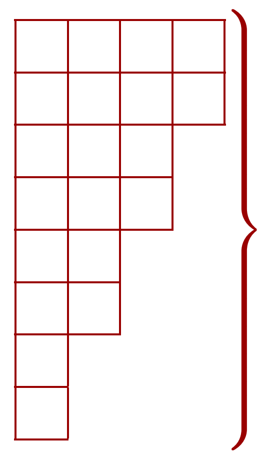
In all that follows, set $q = \mathrm{e}^{2i\pi/3}$.

Okada ('02): $A_n(x_1, \ldots, x_n; y_1, \ldots, y_n)$ is a symmetric function of the full set of parameters x_i, y_i.

$$z_i \equiv x_i \qquad z_{i+n} \equiv y_i \qquad i = 1 \ldots n$$

It is a Schur function: (up to a prefactor)

$$A_n(z_1, \ldots, z_{2n}) = s_Y(z_1, \ldots, z_{2n}) \qquad Y =$$



$2n - 2$

It is entirely characterized by the following properties: (Stroganov, '04)

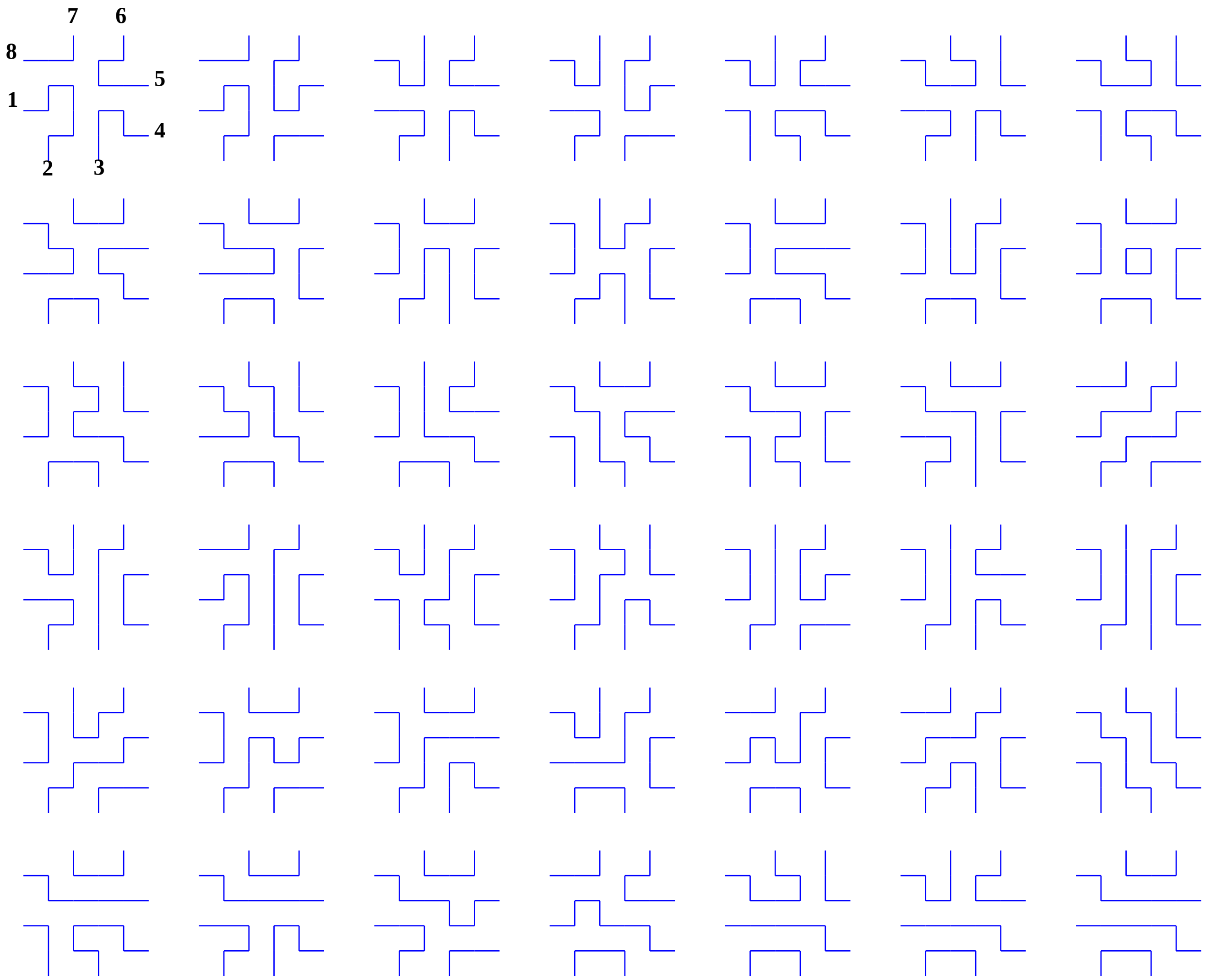
(i) It is a symmetric [homogeneous] polynomial of the z_i, of degree $n-1$ in each variable.

(ii) It satisfies the recursion relation

$$A_n(z_1, \ldots, z_{2n})\Big|_{z_j = q\, z_i} = \prod_{\substack{k=1 \\ k \neq i,j}}^{2n} (q^2 z_i - z_k)\, A_{n-1}(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{j-1}, z_{j+1}, \ldots, z_{2n}) \,.$$

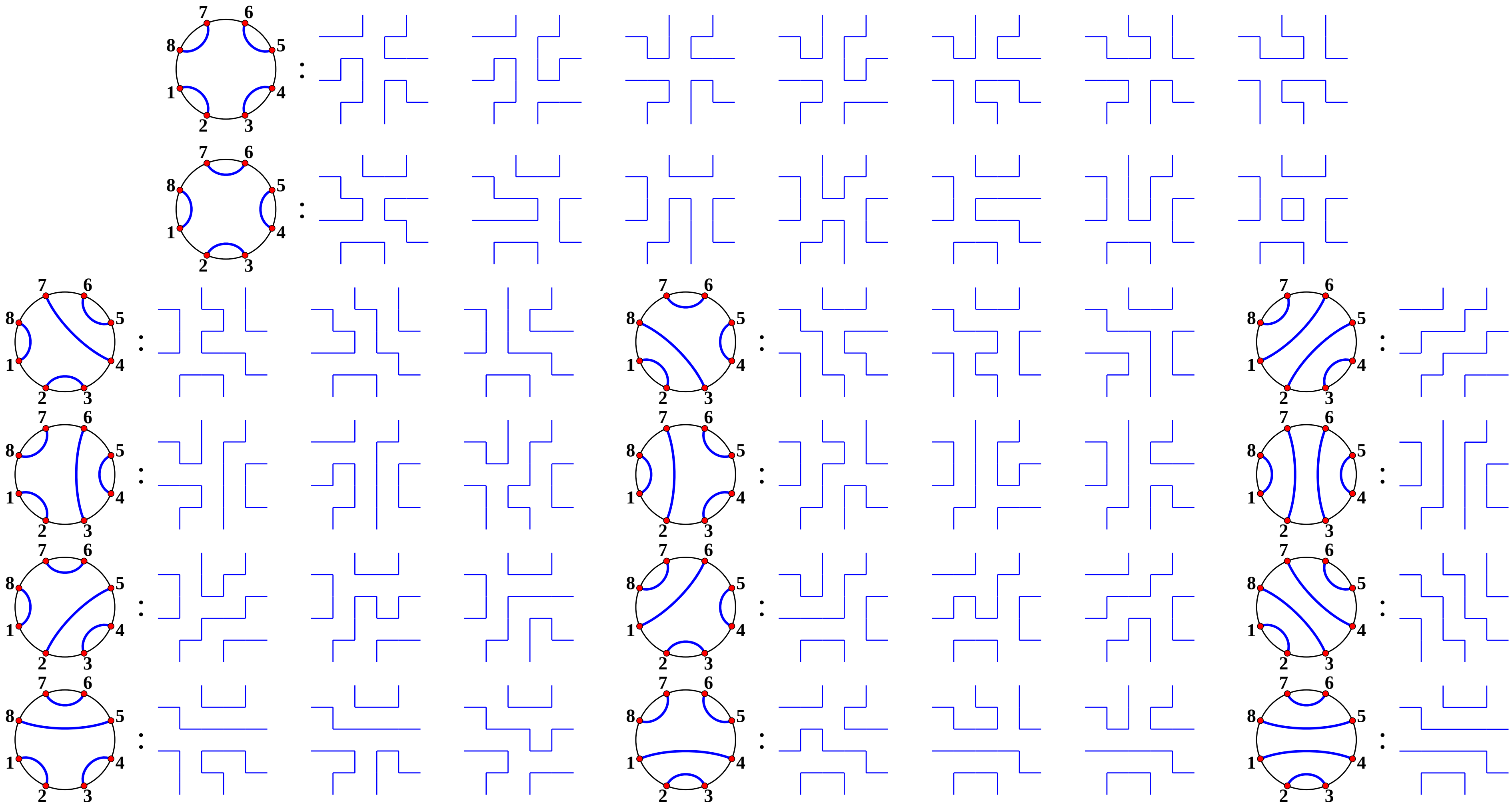
Fully Packed Loops

Example: The 42 FPL on a 4×4 grid



Fully Packed Loops cont'd

FPL configurations fall into Connectivity Classes with a given link pattern π of their external links:



Call $A_n(\pi)$ the FPL configurations with pattern π.

Razumov–Stroganov conjecture

The Perron–Frobenius eigenvector of $T_n(t)$, solution of $T_n(t)\,|\Psi_n\rangle = |\Psi_n\rangle$, is

$$|\Psi_n\rangle = \sum_{\pi} A_n(\pi)\,|\pi\rangle$$

i.e. with proper normalization, its components $\Psi_n(\pi) = A_n(\pi)$ (Razumov & Stroganov '01).

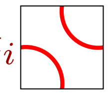
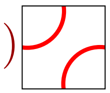
Rk 1: RS implies properties (1), (3) above — (2) not so obvious?

Rk 2: Other types of b.c. on TL $\leftrightarrow$ different symmetry classes of ASM/FPL

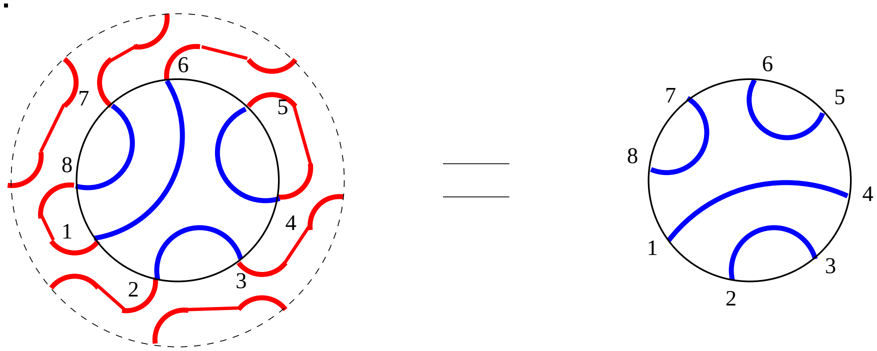
(Batchelor, de Gier & Nienhuis '01; Razumov-Stroganov '01; Pearce, de Gier & Rittenberg '01, ...)

Multi-parameter generalization

Define an **inhomogeneous transfer matrix**: [Di Francesco & PZJ '04]

$$T_n(t; z_1, \ldots, z_{2n}) = \prod_{i=1}^{2n} \Big(t_i$$  $$+ (1 - t_i)$$  $$\Big)$$

with $t_i = \frac{q\, z_i - u}{q\, u - z_i}$, $t = \frac{q-u}{q\, u - 1}$.



Property (Yang–Baxter): $[T_n(t), T_n(t')] = 0$.

The "Perron–Frobenius eigenvector" is

$$T_n(t; z_1 \ldots, z_{2n}) \left|\Psi_n(z_1, \ldots, z_{2n})\right\rangle = \left|\Psi_n(z_1, \ldots, z_{2n})\right\rangle$$

Rk: when all $z_i = 1$, we recover the homogeneous $T_n(t)$, and $\left|\Psi_n\right\rangle$.

What can we say about the components: $\left|\Psi_n(z_1, \ldots, z_{2n})\right\rangle = \sum_\pi \Psi_n(\pi \,|\, z_1, \ldots, z_{2n}) \left|\pi\right\rangle$

and their sum?

Some proved results [Di Francesco & PZJ '04]

⋆ *Polynomiality.*

The components of $\left|\Psi_n(z_1,\ldots,z_{2n})\right\rangle$ are homogenous polynomials of total degree $n(n-1)$ and of partial degree at most $n-1$ in each z_i.

⋆ *Factorization and symmetry.*

$$\Psi_n(\pi\,|\,z_1,\ldots,z_{2n}) = \left(\prod_{s\in E_\pi}\prod_{\substack{i,j\in s\\ i<j}}(q\,z_i - z_j)\right)\Phi_n(\pi\,|\,z_1,\ldots,z_{2n})$$

where $\Phi_n(\pi)$ is a polynomial symmetric in the set of variables $\{z_i, i\in s\}$ for each subset s.

The sum of components is a symmetric polynomial of all z_i.

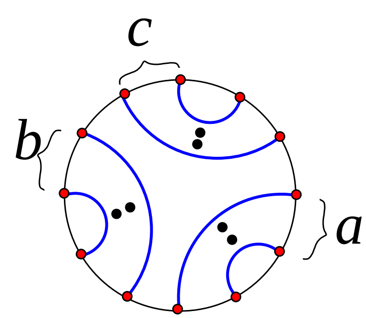
⋆ *Recursion relations.*

Components $\Psi_n(\pi\,|\,z_1,\ldots,z_{2n})$ satisfy linear recursion relations when $z_j = q\,z_i$; in particular, the sum satisfies the Izergin/Stroganov recursion relation, and therefore

$$\sum_\pi \Psi_n(\pi\,|\,z_1,\ldots,z_{2n}) = A_n(z_1,\ldots,z_{2n})$$

Further results

⋆ [PZJ, unpublished] For the link patterns of type (a, b, c):



the components $\Psi_n(\pi \,|\, z_1, \ldots, z_{2n})$ provide a multi-parameter generalization of the Mac-Mahon formula for the number of plane partitions.

⋆ On the other hand, for $z_i = 1$ the number of FPLs with this link pattern is known to coincide with the number of plane partitions [Di Francesco, PZJ & Zuber, '04]. $\Rightarrow$

Theorem [PZJ]: Razumov–Stroganov proved for link patterns of type (a, b, c)

⋆ [Di Francesco, '05] Conjectured results for the multi-parameter open boundary case. Mysterious connection to UASMs/VSASMs...

⋆ What about an integrable model of *crossing* loops? See Knutson's talk...