Physics 209
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Homework - Field Strength Tensor

1 Stress tensor

Recall that the electromagnetic energy density is $u = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2)$.

- Write u in terms of the field-strength tensor $F^{\alpha\beta}$.
- Show that u can be written as the 00 component of a symmetric rank-2 tensor $\Theta^{\alpha\beta}$. I.e., $u = \Theta^{00}$. Find a simple expression for $\Theta^{\alpha\beta}$ in terms of $F^{\alpha\beta}$. (You may lower and raise indices.)
- What is the physical meaning of the component Θ^{10}?

2 Uniform $\vec{E}$ and $\vec{B}$ [*]

A relativistic particle with charge q and mass m is moving in a constant EM field with $\vec{E} = E\hat{x}$ and $\vec{B} = B\hat{z}$. Write down $F^{\alpha\beta}$ and solve the linear differential equation of motion for U^α by diagonalizing $F^\alpha{}_\beta$. Assume that the particle starts from rest at proper time $\tau = 0$. Note that there are three cases: $|E| > |B|$, $|E| < |B|$ and $|E| = |B|$.

3 Polarization tensor [***]

In this problem we will study the microscopic definition of the polarization tensor $S^{\alpha\beta}$.

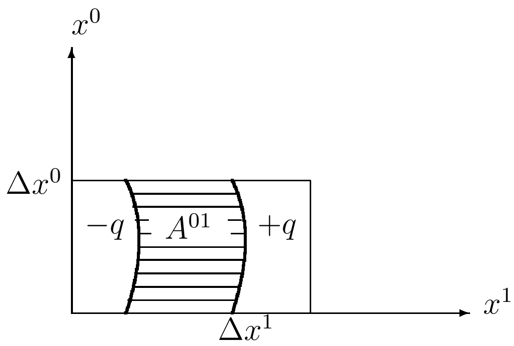
A. Show that the infinitesimal 4-volume element $cdx_1dx_2dx_3dt \equiv dx^0dx^1dx^2dx^3$ is invariant under Lorentz transformations. [**Hint:** write down the Jacobian.]

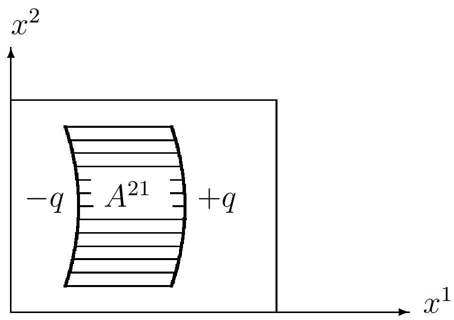
B. Consider a small fluid 4-volume element. Suppose that N particle world-lines cross that 4-volume. Let their worldlines be $x_j^\alpha(\tau_j)$ where τ_j is the proper time of the j^{th} particle and $j = 1 \dots N$. Let Q_j be the charge of the j^{th} particle. Set $\Delta \equiv dx^0dx^1dx^2dx^3$. Show that

$$j^\alpha \Delta = \sum_{j=1}^{N} Q_j \int U_j^\alpha(\tau_j) d\tau_j, \qquad U_j^\alpha \equiv \frac{dx_j^\alpha}{d\tau_j}.$$

where j^α is the 4-current. The integrals on the right-hand-side are over the portion of the world-line that lies inside the 4-volume Δ.

C. Now we would like to find a microscopic expression for the polarization tensor $S^{\alpha\beta}$. We assume that the particles are ordered in pairs of opposite charges q and $-q$. (These charges could represent two opposite ends of an electric dipole, or a positive ion with an orbiting negative charge for a magnetic dipole.)





Define $A^{\alpha\beta}$ as follows. Draw the worldlines of the two oppositely charged particles in the pair. Each particle will enter the 4-volume (which we will call "the box") at one "entry point" and exit at another "exit point." Assume that the particles are sufficiently close to each other so that they will "exit" the box through the same face. Connect the two exit points by a straight line and connect the two "entry" points by a straight line. Now take the projection of these worldlines on the $x^\alpha - x^\beta$ plane. $A^{\alpha\beta}$ is defined as the area bounded between the worldlines. (See the picture for an example of two particles that enter the box at the face given by constant time $x^0 = 0$ and leave through the face given by constant time $x^0 = \Delta x^0$. Their projections on the $x^0 - x^1$ and $x^2 - x^1$ planes are drawn.) $A^{\alpha\beta}$ is defined to be positive if the positively charged particle is to the right of the negatively charged particle worldline when the picture is drawn such that x^α is the vertical axis and x^β is the horizontal axis. Otherwise, $A^{\alpha\beta}$ is defined to be negative. This makes sure that $A^{\alpha\beta} = -A^{\beta\alpha}$. ($A^{01}$ and A^{21} are positive in the picture.)

Show that the polarization tensor $S^{\alpha\beta}$ (containing the electric and magnetic dipole densities) is given by

$$S^{\alpha\beta}\Delta = \sum_{\#} q_{\#} A^{\alpha\beta}_{\#}.$$

where the sum is over all the pairs. Note that if we close the area $A^{\alpha\beta}$ bounded by the two particles with horizontal lines at the top and the bottom to make up a closed curve C then

$$A^{\alpha\beta} = \frac{1}{2}\oint_C [x^\alpha dx^\beta - x^\beta dx^\alpha],$$

where x^α is the 4-coordinate along the curve. This makes it clear that $A^{\alpha\beta}$ is an anti-symmetric tensor.