String Theory 250
Problem Set #4
Feb 21, 2005
Ori Ganor

Conformal Field Theory I

1 Anomalous dimensions

In the free-scalar CFT with action

$$S = \frac{1}{2\pi\alpha'}\int \partial\phi\overline{\partial}\phi d^2 z,$$

the operator $V \equiv: \exp(ik\phi):$ has weights $(h, \tilde{h}) = (\alpha' k^2/4, \alpha' k^2/4)$ and dimension $\Delta \equiv h + \tilde{h} = \alpha' k^2/2$. This is an anomalous dimension which in this exercise we will understand by regularizing.

Set $z = \sigma + i\tau$ and take the range

$$-\infty < \tau < \infty, \qquad 0 \le \sigma \le \ell.$$

replace the segment $[0, \ell]$ by M lattice sites with $\ell = Ma$ and regularize the Lagrangian as

$$L = \frac{1}{4\pi\alpha'}\int_0^\ell d\sigma\,(\partial_\tau\phi\partial_\tau\phi - \partial_\sigma\phi\partial_\sigma\phi) \longrightarrow \frac{1}{4\pi\alpha'}\left[a\sum_{r=1}^{M}\dot{\phi}_r^2 - \frac{1}{a}\sum_{r=1}^{M-1}(\phi_r - \phi_{r+1})^2\right].$$

This problem is usually solved in solid state physics. The eigen-frequencies are

$$\omega_j = \frac{2}{a}\sin\frac{j\pi}{2M}, \qquad j = 0, \dots, M-1.$$

Compare the operator $\exp(ik\phi_r)$ to the normal ordered operator $:\exp(ik\phi_r):$ [you may normal-order by moving all creation operators to the left and annihilation operators to the right], and show that $a^{-\alpha' k^2/2}\exp(ik\phi_r)$ is finite in the limit $a \to 0$. Deduce the anomalous dimension $\Delta = \alpha' k^2/2$. You will have to use

$$\frac{Ma}{\pi}\sum_j \frac{1}{\omega_j} = \log M + O(1).$$

2 Conformal symmetry in $D > 2$

The conformal symmetry group in $D > 2$ is $SO(D+1, 1)$. These are transformations of the form

$$x^i \to u^i + \theta\frac{\Lambda^i{}_j(v^j + x^j)}{\|\vec{v} + \vec{x}\|^2},$$

where Λ is an $SO(D)$ matrix, θ is a nonzero scalar, and $\vec{u}$ and $\vec{v}$ are vectors. We will now study this group of transformations.

- By checking how the metric transforms show that these are conformal transformations.

- The identity transformation is obtained by taking $\Lambda^i{}_j = \delta^i{}_j$, $\theta = \|\vec{v}\|^2 \to \infty$ and $\vec{u} = -\vec{v}$. Show that an infinitesimal conformal transformation takes the form
$$\delta x^i = \epsilon(a^i + b^i{}_j x^j + f x^i + c_j x^j x^i - \frac{1}{2}\vec{x}^2 c^i),$$
where $b_{ij} = -b_{ji}$ is an antisymmetric matrix, $\vec{a}, \vec{c}$ are vectors, and f is a scalar.

- Let $P^i, J^i{}_j, S, K_j$ be the operators that generate infinitesimal transformations corresponding to $a^i, b^i{}_j, f, c_j$, respectively. Find their commutation relations.

- Show that the above are all the solutions to the conformal Killing equation
$$\partial_i \delta x_j + \partial_j \delta x_i = \frac{2}{D}\delta_{ij}\partial_k \delta x_k.$$

3 Schwarzian derivative

In this exercise we will derive the expression for the Schwarzian derivative $\{f, z\}$ from its **infinitesimal form**:
$$\{f(z), z\} = \epsilon v'''(z) + O(\epsilon)^2, \qquad f(z) = z + \epsilon v(z).$$

- **Behavior under composition:** From the transformation properties of the energy momentum tensor $T(z)$, show that the Schwarzian derivative has to satisfy
$$\{f(g(z)), z\} = \{g(z), z\} + g'(z)^2\{f(g(z)), g(z)\}.$$

- Assuming that $\{f, z\}$ is an algebraic expression in f and its derivatives, and assuming the infinitesimal form of the Schwarzian derivative and its behavior under composition, show that the Schwarzian derivative has to take the form
$$\{f, z\} = A(f', f'')f''' + B(f', f''),$$
where A and B are algebraic expressions.

- From the infinitesimal form of the Schwarzian derivative and its behavior under composition, show that the Schwarzian derivative of a Möbius ($PSL(2, \mathbf{C})$) transformation
$$f(z) = \frac{az + b}{cz + d}, \qquad ad - bc = 1,$$
is zero.

- Using the above fact about Möbius transformations, find $B(f', f'')$.

- Using the above fact about Möbius transformations and the behavior of the Schwarzian derivative under composition, show that
$$\{\frac{ag(z) + b}{cg(z) + d}, z\} = \{g(z), z\},$$
and use this to determine $A(f', f'')$.

Is there a generalization of the Schwarzian derivative that satisfies
$$\{f(g(z)), z\} = \{g(z), z\} + g'(z)^n\{f(g(z)), g(z)\}, \qquad \{z + \epsilon u(z), z\} = \epsilon u^{(n+1)}(z) + O(\epsilon^2)?$$