String Theory 250
Problem Set #5
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Conformal Field Theory II

1 Correlation functions of $T(z)$

1. Let $\mathcal{O}_1, \ldots, \mathcal{O}_n$ be n primary fields with weights $(h_i, \tilde{h}_i)$ $[i = 1, \ldots, n]$. Use the OPE of $T(z)$ with $\mathcal{O}_i(z_i)$ to show that

$$\langle T(z)\mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\rangle_{S^2} = \sum_{i=1}^{n}\left[\frac{h_i}{(z-z_i)^2} + \frac{1}{(z-z_i)}\frac{\partial}{\partial z_i}\right]\langle\mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\rangle_{S^2}$$

[See equation (15.2.1) of Polchinski II.]

2. Let $L_{-k}\cdot\mathcal{O}_1$ be a descendent of the primary operator $\mathcal{O}_1$. Use the previous result to show that

$$\langle L_{-k}\mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\rangle_{S^2} = \sum_{i=2}^{n}\left[\frac{h_i(k-1)}{(z_i-z_1)^k} + \frac{1}{(z_i-z_1)^{k-1}}\frac{\partial}{\partial z_i}\right]\langle\mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\rangle_{S^2}$$

[See equation (15.2.3) of Polchinski II.]

2 Free fermions

A chiral fermion $\psi(z)$ has conformal weights $h = 1/2$ and $\tilde{h} = 0$. The energy momentum tensor is $T(z) = -(1/2)\psi\partial\psi$.

1. Use the OPE $\psi(z)\psi(0) \sim 1/z$ to find the central charge of the system.

2. The mode expansion is

$$\psi(z) = \sum_{n=-\infty}^{\infty}\frac{\psi_{n-\frac{1}{2}}}{z^n}.$$

Why is the mode number half an integer and not a full integer?

3. The ground state of the Hilbert space is defined by

$$\psi_{n+\frac{1}{2}}|0\rangle = 0, \qquad n = 0, 1, 2, \ldots$$

Find the states that correspond to the local operators $1, \psi(z)$ and $T(z)$.

4. Using the Hilbert space approach, calculate the correlation functions

$$\langle\psi(0)T(z)\psi(\infty)\rangle_{S^2}, \langle\psi(0)\psi(z_1)\psi(z_2)\psi(\infty)'\rangle_{S^2}.$$

(Here $\psi(\infty)' = \lim_{z\to\infty} z\psi(z)$, as is needed in order to get a finite answer.) Repeat the calculation using just the $\psi\psi$ and $T(z)\psi$ OPEs together with holomorphicity arguments. [Assume that $\langle 1\rangle_{S^2}$ is normalized to 1.]

5. Find a constant a such that the descendent $(L_{-2} + aL_{-1}^2) \cdot \psi(z)$ vanishes. Using problem (1), this implies a certain 2^{nd} order differential equation for the correlator

$$\langle \psi(0)\psi(z)\psi(1)\psi(\infty)' \rangle_{S^2}.$$

Write it down, and check that it is satisfied.

6. Now suppose that we take the modes to be ψ_n with an integer n (instead of the previous $\psi_{n+\frac{1}{2}}$). The expansion of the field is

$$\psi(z) = \sum_{n=-\infty}^{\infty} \frac{\psi_n}{z^{n+\frac{1}{2}}}.$$

There are now two ground states $|\uparrow\rangle$ and $|\downarrow\rangle$ (why?). Show that both of them have conformal weight $h = 1/16$.