String Theory 250
Jan 25, 2005
Ori Ganor

Problem set # 1

1 Polyakov action

Derive the expression

$$\delta_\gamma S_P[X,\gamma] = -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma (-\gamma)^{\frac{1}{2}} \delta\gamma^{ab}(h_{ab} - \frac{1}{2}\gamma_{ab}\gamma^{cd}h_{cd}), \qquad [1.2.14,\ p12]$$

for the variation $\delta_\gamma S_P[X,\gamma]$ of the Polyakov action under a change in the worldsheet metric.

2 2D differential geometry

Define coordinates

$$\sigma_\pm = \sigma \pm \tau.$$

Find the Christofel symbols Γ^a_{bc}, Curvature tensor R_{abcd}, Ricci tensor R_{ab}, Ricci scalar R, and Einstein tensor $R_{ab} - \frac{1}{2}\gamma_{ab}R$ for the metric

$$ds^2 = e^\phi d\sigma_+ d\sigma_-$$

where ϕ is a function of σ_+, σ_-. Explain what your result for the Einstein tensor means physically.

3 Problem 1.3 of [1] [*]

For worldsheets with boundary, show that

$$\chi = \frac{1}{4\pi}\int_M d\tau d\sigma (-\gamma)^{\frac{1}{2}} R + \frac{1}{2\pi}\int_{\partial M} k\, ds,$$

where ∂M is the boundary curve, ds is the proper time in the boundary metric γ_{ab} and k is the geodesic curvature of the boundary,

$$k = \pm t^a n_b \nabla_a t^b,$$

where t^a is a unit vector tangent to the boundary and n^a is an outward pointing unit vector orthogonal to t^a. The upper sign is for a timelike boundary and the lower sign for a spacelike boundary.

4 3D soap bubbles [*]

It is often useful to consider the analog of the Polyakov action in Euclidean target space. In this case it is convenient to write (σ_1, σ_2) instead of (τ, σ). The classical solutions are surfaces of minimal area. In this problem we will apply this technique to three-dimensional soap bubbles.

[*] Consider a parameterization

$$X^1 = \sigma_1, \qquad X^2 = \sigma_2, \qquad X^3 = f(\sigma_1, \sigma_2).$$

Using the Nambo-Goto action, find a differential equation for $f(\sigma_1, \sigma_2)$.

- If we have cylindrical symmetry, we can write f as a function of the radius $r \equiv \sqrt{\sigma_1^2 + \sigma_2^2}$. Write an ordinary differential equation for $f(r)$ and solve it.
- Now use the Polyakov action to find the general solution $f(r)$.

[***] Use the Polyakov action to find the shape of a soap bubble suspended between the infinite line $X^2 = 0, X^3 = -a$ and the line $X^1 = 0, X^3 = a$. **Hint:** A harmonic function on the (σ^1, σ^2) plane can be expressed as the real part of an analytic function in $z \equiv \sigma^2 + i\sigma^1$. Try to find the derivatives of the analytic functions corresponding to X^1, X^2, X^3. As a further hint, take the worldsheet to be a strip given by

$$-\frac{\pi}{2} \leq \sigma^1 \leq \frac{\pi}{2}, \qquad -\infty < \sigma^2 < \infty$$

and look for a solution which is invariant under the symmetry

$$X^2(\sigma^1, \sigma^2) \longrightarrow X^1(-\sigma^1, \sigma^2), \quad X^1(\sigma^1, \sigma^2) \longrightarrow X^2(-\sigma^1, \sigma^2), \quad X^3(\sigma^1, \sigma^2) \longrightarrow -X^3(-\sigma^1, \sigma^2).$$

5 Holomorphic curves

Assume that both the worldsheet as well as the target space have metrics of Euclidean signature. Suppose the taget space is even dimensional $D = 2n$. Define complex valued fields

$$Z^1 \equiv X^1 + iX^{1+n}, \quad Z^2 \equiv X^2 + iX^{2+n}, \cdots, Z^n \equiv X^n + iX^{2n}.$$

Also define $z \equiv \sigma^1 + i\sigma^2$ for the worldsheet coordinate. Take a classical configuration with $Z^j(z)$ all given by holomorphic functions. In algebraic geometry, this is called a *holomorphic curve*. Show that a holomorphic curve is a solution to the equations of motion derived from the Polyakov action. Holomorphic curves are therefore minimal area surfaces.

References

[1] J. Polchinski, "String Theory," Cambridge University Press.

[2] M. B. Green, J. H. Schwarz and E. Witten, "Superstring Theory," Cambridge University Press.