Physics 505
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Ori Ganor

Advanced topics in ordinary differential equations

13.1 Γ-functions

The Γ-function is a generalization of the factorial to complex numbers. $\Gamma(z)$ is defined for complex numbers z such that for positive integers: $\Gamma(n) = (n-1)!$. For $\Re z > 0$ it is defined as:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

For $\Re z \leq 0$ we have to use analytic continuation or the formula:

$$\Gamma(z) = \frac{1}{z}\Gamma(z+1).$$

This formula can be used several times so that:

$$\Gamma(z) = \frac{1}{z(z+1)\cdots(z+n-1)}\Gamma(z+n).$$

This formula shows that $\Gamma(z)$ has simple poles for $z = -n$ where n is a nonnegative integer. Note that:

$$\Gamma(\frac{1}{2}) = \int_0^\infty \frac{1}{\sqrt{t}} e^{-t} dt = \sqrt{\pi}.$$

13.2 Confluent hypergeometric functions

The confluent hypergeometric function is a holomorphic function defined by:

$$F(\alpha, \gamma, z) = 1 + \frac{\alpha}{\gamma}\frac{z}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{z^2}{2!} + \cdots$$

Note that it converges for all finte z. The parameter α is arbitrary while γ is not allowed to be zero or a negative integer. Note also that $F(\alpha, \alpha, z) = e^z$ and the relation:

$$F(\alpha, \gamma, z) = e^z F(\gamma - \alpha, \gamma, -z).$$

The function $\psi(z) \equiv F(\alpha, \gamma, z)$ satisfies the differential equation:

$$z\psi'' + (\gamma - z)\psi' - \alpha\psi = 0.$$

The second independent solution of this equation is given by: $z^{1-\gamma}F(\alpha - \gamma + 1, 2 - \gamma, z)$, as is easily checked. So the general solution is of the form:

$$\psi = C_1 F(\alpha, \gamma, z) + C_2 z^{1-\gamma} F(\alpha - \gamma + 1, 2 - \gamma, z).$$

We sometimes need to know the asymptotic expansion for large z. It is:

$$\begin{aligned} F(\alpha, \gamma, z) &= \frac{\Gamma(\gamma)}{\Gamma(\gamma - \alpha)}(-z)^{-\alpha} G(\alpha, \alpha - \gamma + 1, -z) + \frac{\Gamma(\gamma)}{\Gamma(\alpha)} e^z z^{\alpha - \gamma} G(\gamma - \alpha, 1 - \alpha, z), \\ G(\alpha, \beta, z) &= 1 + \frac{\alpha\beta}{1!\, z} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\, z^2} + \cdots \end{aligned}$$

For α that is a negative integer we have the equation:

$$F(-n, \gamma, z) = \frac{1}{\gamma(\gamma+1)\cdots(\gamma+n-1)} z^{1-\gamma} \frac{d^n}{dz^n}(e^{-z} z^{\gamma+n-1}).$$

13.3 Application to the Coulomb field

The confluent hypergeometric function appears in the solution of Schrödinger's equation in a Coulomb potential. The radial wave-function of the continuous spectrum is:

$$R_{kl}(r) = \frac{C_k}{(2l+1)!}(2kr)^l e^{-ikr} F(\frac{imZe^2}{\hbar^2 k} + l + 1, 2l + 2, 2ikr),$$

where C_k are normalization factors. If we take

$$C_k = \sqrt{\frac{2}{\pi}} k e^{\frac{\pi}{2k}} \left| \Gamma(l + 1 - \frac{imZe^2}{\hbar^2 k}) \right|,$$

then the asymptotic behavior of R_{kl} can be calculated as:

$$R_{kl} \approx \sqrt{\frac{2}{\pi}} \frac{1}{r} \sin(kr + \frac{mZe^2}{\hbar^2 k} \log 2kr - \frac{1}{2} l\pi + \delta_l),$$

where:

$$\delta_l = \arg\Gamma(l + 1 - \frac{imZe^2}{\hbar^2 k}).$$

Note that this is not quite the standard asymptotic behavior of a radial wave-function. The factor $\log 2kr$ does not appear in the usual expansion. The Coulomb force does not decay fast enough as $r \to \infty$ to have the usual asymptotic behavior. But since the only correction is a term that is independent of l, we can use δ_l as the phase-shifts in our scattering formulas.

For more details see Landau& Lifshitz appendix d.