M-THEORY AND E_{10}

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BASED ON:

- "M-Theory and E_{10}: Billiards, Branes and Imaginary Roots," [arXiv:hep-th/0401053], Jeff Brown, Craig Helfgott and OG

- "E_{10}-Orbifolds," [arXiv:hep-th/0409037], Jeff Brown, Surya Ganguli, Craig Helfgott, OG

Introduction

In this talk we will take a different look at compactified M-theory.

We will expand on a relation between M-theory and the mysterious infinite-dimensional Lie algebra E_{10}.

Main new points:

- Classification of fluxes and "charges" in terms of roots of E_{10}.
  (A U-duality invariant generalization of K-theory?)

- Masses of Kaluza-Klein particles and branes in terms of "E_{10} variables."

- Beyond classical supergravity: attempt to begin constructing a Hamiltonian and Hilbert space as Quantum Mechanics on a coset of E_{10}.

- Imaginary roots of E_{10} play a crucial role!

Motivation

Our setting is M-theory on "T^{10}."

- When all spatial dimensions are compact there are no boundary conditions that can fix the compactification parameters.

  (We need at least two noncompact spatial directions to have *moduli*.)

- Even the topology of T^{10} is not fixed.

- Old conjecture: the infinite dimensional Lie algebra E_{10} plays a role at the level of classical supergravity. Julia, '85

Can E_{10} describe branes?

Do branes carry "E_{10}-charges?"

Can we use E_{10} to describe brane interactions?

The generators of E_{10}

Consider the classical, dimensionally reduced, Kasner metric on T^{10}:

$$ds^2 = -dt^2 + \sum_{i=1}^{10} R_i(t)^2 dx_i^2, \qquad G = dC = 0.$$

$$0 \leq x_i \leq 2\pi, \quad i = 1 \ldots 10.$$

The generators of E_{10} can be split into:

10	C.S.A.	rescaling	$R_i \to e^{\epsilon} R_i$
∞	real roots	turn on fluxes	$C \to C + \epsilon, \cdots$
∞	imaginary roots	?	?

Problems

We immediately encounter several problems if we try to interpret E_{10} at a quantum level:

- E_{10} is explicitly broken in perturbation theory and by instanton effects. For example: obviously $R_i \to e^{\epsilon} R_i$ cannot be an exact symmetry!

- The original proposal of E_{10} is supposed to describe only the dimensionally reduced classical SUGRA. None of the fields is allowed to vary in space.

- Equivalently, where are the Kaluza-Klein particles in this formalism?

  Where are the M2-branes, M5-branes, etc.?

- What do the mysterious generators that are related to imaginary roots do?

- What are the Hilbert space and Hamiltonian on which E_{10} acts?

Our proposal

Real roots describe fluxes.

Imaginary roots (with certain additional properties) describe brane-charges.

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It has to be mentioned that imaginary roots also have another role. They parameterize classical derivative fields in a "small tension expansion" of the supergravity effective action.
Damour, Henneaux and Nicolai 2002

We will not discuss this role here.

The Lie Algebra E_{10}

What do the generators of E_{10} look like?

10	C.S.A.	$J^i \ (i = 1, \ldots, 10)$
∞	positive real roots α	$J^{+\alpha}$
∞	positive imaginary roots	$J^{+\gamma,j}$
∞	negative roots	$J^{-\alpha}, J^{-\gamma,j}$

10	C.S.A.	$\Rightarrow$ Radii R_i
∞	positive real roots α	$\Rightarrow$ fluxes, angles, ...
∞	positive imaginary roots	$\Rightarrow$? (branes)
∞	negative roots	$\Rightarrow$ ("gauge fixed")

Let's describe the indices α, γ that label $J^{+\alpha}, J^{+\gamma,j}$.

The root lattice

The E_{10} generators $J^{+\alpha}, J^{+\gamma,j}$ are labeled by 10 integers,

$$\alpha = (n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9, n_{10}).$$

The integers are required to satisfy

$$n_i \in \mathbf{Z}, \quad (i = 1 \ldots 10), \qquad \sum_{i=1}^{10} n_i \equiv 0 \bmod 3,$$

These conditions define the root lattice.

There are two extra conditions that α must satisfy. The first is

$$2 \geq \alpha^2 = \sum_{i=1}^{10} n_i^2 - \frac{1}{9} \left(\sum_{i=1}^{10} n_i \right)^2$$

All the conditions so far define the root space.

The last condition is that the first nonzero n_i should be positive.

Such α's are called positive roots.

Example: $SL(3,\mathbb{R})$

Before we move on, let us demonstrate the analogs of J^α for a finite Lie group such as $SL(3,\mathbb{R})$.

Every element of $SL(3,\mathbb{R})$ can be uniquely written as

$$\begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix} \begin{pmatrix} e^{h_1} & & \\ & e^{h_2} & \\ & & e^{-h_1-h_2} \end{pmatrix} \underbrace{\begin{pmatrix} \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{pmatrix}}_{\in SO(3)}$$

In this case we would write

$$J^{(1,0)} \to \begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{pmatrix}, \qquad J^{(1,1)} \to \begin{pmatrix} 0 & & 1 \\ & 0 & \\ & & 0 \end{pmatrix},$$

and

$$J^{(0,1)} \to \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & & 0 \end{pmatrix}.$$

We only have three different labels, in this case.

Roots and Euclidean branes

What is the connection to the physics of M-theory?

$$ds^2 = -dt^2 + \sum_{i=1}^{10} R_i(t)^2 dx_i^2, \qquad G = dC = 0.$$

Set

$$h_i = \log[M_p R_i], \qquad i = 1, \ldots, 10.$$

The quantity

$$\frac{1}{2\pi} S_\alpha = e^{\langle \vec{h}, \alpha \rangle} \equiv e^{n_1 h_1 + \cdots + n_{10} h_{10}} = \prod_{i=1}^{10} (M_p R_i)^{n_i}$$

will play a central role in what follows.

Examples

$$\frac{1}{2\pi} S_\alpha = e^{\langle \vec{h}, \alpha \rangle} \equiv e^{n_1 h_1 + \cdots + n_{10} h_{10}} = \prod_{i=1}^{10} (M_p R_i)^{n_i}$$

Let us take a few examples of allowed roots:

$$\alpha = (0,0,0,0,0,0,0,0,1,-1) \Rightarrow e^{\langle \vec{h}, \alpha \rangle} = \frac{R_9}{R_{10}},$$

$$\alpha = (0,0,0,0,0,0,0,1,1,1) \Rightarrow e^{\langle \vec{h}, \alpha \rangle} = M_p^3 R_8 R_9 R_{10},$$

$$\alpha = (0,0,0,0,1,1,1,1,1,1) \Rightarrow e^{\langle \vec{h}, \alpha \rangle} = M_p^6 R_5 \cdots R_{10},$$

$$\alpha = (0,0,1,1,1,1,1,1,1,2) \Rightarrow e^{\langle \vec{h}, \alpha \rangle} = M_p^9 R_3 \cdots R_9 R_{10}^2$$

These are the actions of: a Euclidean Kaluza-Klein particle, M2-brane, M5-brane, and Kaluza-Klein monopole.

Euclidean branes and fluxes

The Euclidean branes above are instantons.

In a Minkowski metric, they should be interpreted as processes that change a certain flux by one unit.

<u>Example</u>

$$2\pi e^{\langle \vec{h},\alpha\rangle} = 2\pi M_p^3 R_1 R_2 R_3.$$

The relevant flux is $G_{0123} = 4\partial_{[0}C_{123]}$.

We identify $\mathcal{C}_\alpha = (2\pi)^2 C_{123}$.

If the instanton occurs at time t_α then

$$G_{0123} = \begin{cases} N & \text{units, for} \quad t < t_\alpha \\ N+1 & \text{units, for} \quad t > t_\alpha \end{cases}$$

The contribution of this instanton to the amplitude is

$$\sim e^{-2\pi e^{\langle \vec{h},\alpha\rangle} + 2\pi i \mathcal{C}_\alpha} = e^{-2\pi M_p^3 R_1 R_2 R_3 + (2\pi)^3 i C_{123}}.$$

Real and imaginary roots

Recall the definition of the norm

$$\alpha^2 \equiv \sum_{i=1}^{10} n_i^2 - \frac{1}{9}\left(\sum_{i=1}^{10} n_i\right)^2 .$$

For all the roots in the previous example $\alpha^2 = 2$.

The roots of E_{10} fall into two classes:
(V. Kac, *Infinite Dimensional Lie Algebras.*)

$\alpha^2 = 2$	real roots
$\alpha^2 \leq 0$	imaginary roots

In many respects, real roots behave similarly to their counterparts of finite dimensional Lie algebras.

For example, there is one Lie algebra generator $J^{+\alpha}$ for every root
$\Rightarrow$ Real roots have multiplicity $m = 1$.

Imaginary roots behave differently. For example, they can have multiplicity $m > 1$.

This is why we used the notation $\mathcal{C}_{\gamma,j}$ $(j = 1 \ldots m)$.

Classical description - I

Kasner metric on T^{10}

$$ds^2 = -dt^2 + \sum_{i=1}^{10} R_i(t)^2 dx_i^2,$$

$$0 \leq x_i \leq 2\pi, \quad i = 1 \ldots 10,$$

We would like to rewrite
the Einstein-Hilbert action $\int \sqrt{g}\mathcal{R}$.

Define

$$\vec{h} \equiv (h_1, h_2, \ldots, h_{10}), \qquad h_i \equiv \log[M_p R_i].$$

and define conformal time

$$\tilde{\tau} \equiv \int_{t_0}^{t} \frac{dt'}{M_p^9 V_{10}(t')}, \qquad V_{10} \equiv R_1 \cdots R_{10}.$$

Classical description - II

Define

$$\left\|\frac{d\vec{h}}{d\tilde{\tau}}\right\|^2 \equiv \sum_{i=1}^{10}\left(\frac{dh_i}{d\tilde{\tau}}\right)^2 - \left(\sum_{i=1}^{10}\frac{dh_i}{d\tilde{\tau}}\right)^2 .$$

Then

$$\int \sqrt{g}\mathcal{R} d^{10}x dt \longrightarrow \int \left\|\frac{d\vec{h}}{d\tilde{\tau}}\right\|^2 d\tilde{\tau}.$$

The metric $\|\cdot\|^2$ has signature $(9,1)$.

Connection with E_{10}:

Space of $\vec{h} \Longrightarrow \widehat{\mathcal{H}}_{\mathbf{R}}$ Cartan subalgebra of E_{10}.

Classical solution - I

The classical GR equations of motion are

$$\frac{d^2\vec{h}}{d\tilde{\tau}^2} = 0, \qquad \left\| \frac{d\vec{h}}{d\tilde{\tau}} \right\|^2 = 0.$$

The solution is

$$\frac{dh_i}{d\tilde{\tau}} = k_i, \quad (i = 1\ldots 10), \qquad \sum_1^{10} k_i^2 - \left(\sum_1^{10} k_i\right)^2 = 0,$$

To express in terms of t, we recall $M_p dt = M_p^{10} V_{10} d\tilde{\tau}$

We need to calculate V_{10}

$$\frac{d\log[M_p^{10} V_{10}]}{d\tilde{\tau}} = \sum_i \frac{dh_i}{d\tilde{\tau}} = \sum_i k_i$$

Then

$$M_p dt = M_p^{10} V_{10} d\tilde{\tau} = \frac{d\exp\left[\left(\sum k_i\right)\tilde{\tau}\right]}{\sum k_i}$$

$$\Longrightarrow \tilde{\tau} = \frac{\log[M_p t]}{\sum k_i} + \text{const}$$

Classical solution - II

We can express the solution as

$$h_i = k_i\tilde{\tau} + \text{const} = p_i \log[M_p t] + \text{const}\,, \qquad p_i \equiv \frac{k_i}{\sum k_i}.$$

We get the Kasner solution

$$R_i \sim t^{p_i}, \qquad \sum p_i = \sum p_i^2 = 1.$$

The evolution of the universe is described by an abstract particle moving at constant velocity (w.r.t. conformal time $\tilde{\tau}$) in h_i-space $\widehat{\mathcal{H}}_{\mathbf{R}}$.

Asymptotic regions

From that Kasner solution

$$R_i \sim t^{p_i}, \qquad \sum p_i = \sum p_i^2 = 1.$$

We see that

Directions with $p_i > 0$ Expanding
Directions with $p_i < 0$ Contracting

$$\sum p_i = \sum p_i^2 = 1 \Longrightarrow$$

at least one $p_i > 0$ and at least one $p_j < 0$.

But,

$$0 > \sum p_i^2 - \left(\sum p_i\right)^2 \Longrightarrow$$

there exists some (M, IIA, IIB) weakly coupled SUGRA description for $t \to \infty$.

Banks & Fischler & Motl, 1998

We need to add matter for this condition to hold.

Adding fluxes

To start, suppose we turn on only the component G_{1234}. Flux quantization requires it to be an integer. It then contributes a potential term to the classical supergravity action proportional to

$$\sqrt{g}|G|^2 = \frac{G_{1234}^2}{(R_1 R_2 R_3 R_4)^2} V_{10} = \frac{G_{1234}^2}{V_{10}} (R_5 \cdots R_{10})^2,$$

$$V_{10} \equiv R_1 \cdots R_{10}.$$

Switching to conformal time,

$$\tilde{\tau} \equiv \int_{t_0}^{t} \frac{dt'}{M_p^9 V_{10}(t')},$$

The full Lagrangian becomes

$$L = 2\pi \left\| \frac{d\vec{h}}{d\tilde{\tau}} \right\|^2 - \pi [\underbrace{(2\pi)^3 G_{1234}}_{\text{integer}}]^2 e^{2(h_5+h_6+h_7+h_8+h_9+h_{10})}$$

In terms of roots

$$e^{2(h_5+h_6+h_7+h_8+h_9+h_{10})} = e^{2\langle \vec{h}, \alpha \rangle}$$

where

$$\alpha = (0,0,0,0,1,1,1,1,1,1).$$

is a real root.

Adding fluxes - conclusion

To add N units of flux,
augment the Lagrangian by

$$-\pi N^2 e^{2\langle \vec{h}, \alpha \rangle},$$

where α is the real root that corresponds to the flux.

We have the relation:

Real roots $\alpha \Longleftrightarrow$ fluxes $\Longleftrightarrow$ instantons.

Imaginary roots

We have seen that real roots α are related to instantons or fluxes. What is the interpretation of imaginary roots?

Example 1

$$e^{\langle\gamma,\vec{h}\rangle} = M_p^9 R_1 R_2 R_3 R_4 R_5 R_6 R_7 R_8 R_9, \quad \gamma^2 = 0.$$

Recall conformal time $d\tilde{\tau} = dt/M_p^9 V_{10}$. Note that

$$\frac{e^{\langle\gamma,\vec{h}\rangle}}{M_p^9 V_{10}} = (R_{10})^{-1},$$

Could this γ be related to a Kaluza-Klein particle?

Example 2

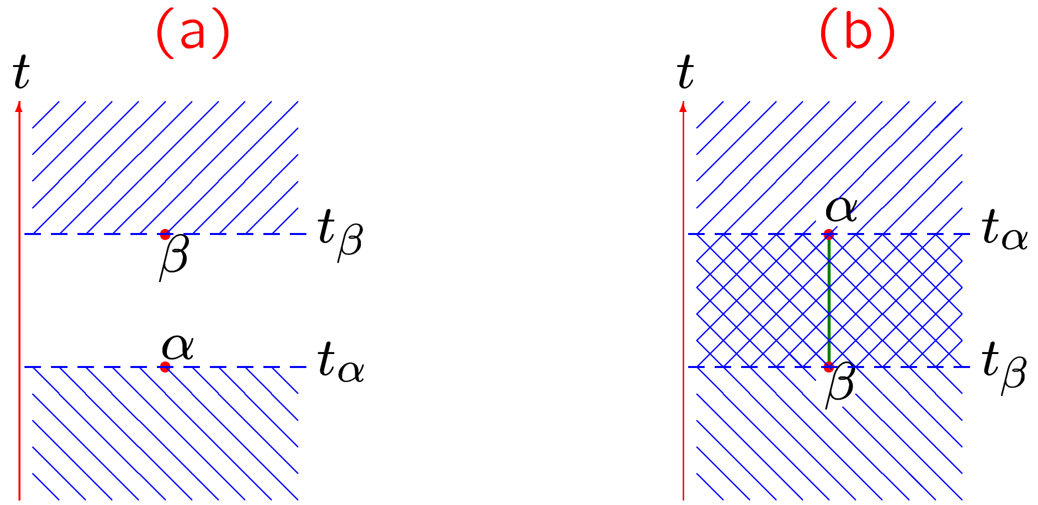
$$e^{\langle\gamma,\vec{h}\rangle} = M_p^{12} (R_1 R_2)^2 R_3 R_4 R_5 R_6 R_7 R_8 R_9 R_{10}, \quad \gamma^2 = 0.$$

Note that

$$\frac{e^{\langle\gamma,\vec{h}\rangle}}{M_p^9 V_{10}} = M_p^3 R_1 R_2,$$

Could this γ be related to an M2-brane?

Brane creation between two instantons

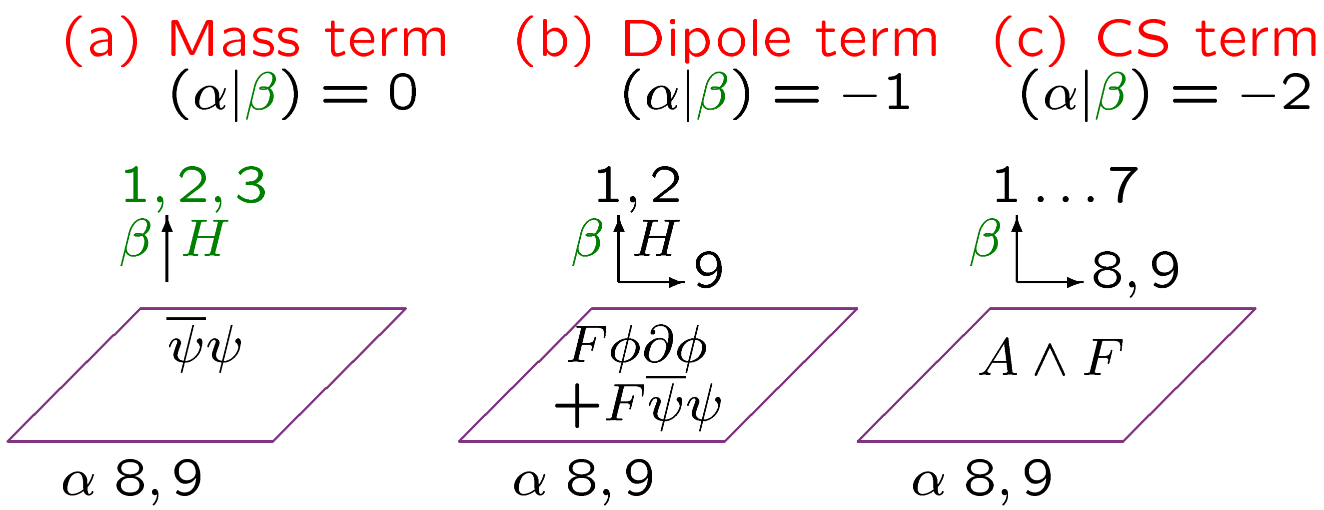


Two instantons associated with the real roots α, β. Each instanton creates a jump in the associated flux. Instanton α creates a jump from a nonzero value to 0, while instanton β changes another flux from 0 to a nonzero value.

(a) Instanton α occurs before instanton β, and the different fluxes do not overlap;

(b) Instanton α occurs after instanton β, and the fluxes overlap between t_α and t_β. In addition a particle associated to $\gamma = \alpha + \beta$ is created between the two instantons.

Interactions of branes with fluxes



The D2-brane is in the plane of the $8^{th}, 9^{th}$ directions (and time). The imaginary root associated with it is α. The flux is associated with the real root β.

The arrows indicate the directions of the flux:

(a) A mass term appears as a result of an NSNS flux orthogonal to the brane;

(b) A dipole interaction appears as a result of an NSNS flux with two legs orthogonal to the brane and one leg parallel to the brane;

(c) A Chern-Simons term appears in massive type-IIA theory (the flux permeates throughout space);

Interactions of branes with fluxes - Summary

To summarize, we have found the following interactions of fluxes with branes

$$
\begin{aligned}
(\alpha|\beta) = 0 \quad &\Longrightarrow \quad \text{Mass term,} \\
(\alpha|\beta) = -1 \quad &\Longrightarrow \quad \text{Dipole interaction,} \\
(\alpha|\beta) = -2 \quad &\Longrightarrow \quad \text{Chern-Simons.}
\end{aligned}
$$

$\mathbf{Z}_2$-orbifolds

Consider a $\mathbf{Z}_2$ orbifold of M-theory on T^{10}, where $\mathbf{Z}_2$ acts as an isometry and, possibly, worldvolume orientation reversal.

We would like to describe this $\mathbf{Z}_2$ action in terms of E_{10}.

Type	Background	Object	Number
IA	$T^8 \times (S^1/\mathbf{Z}_2)$	D8-branes	16
M	$T^5 \times (T^5/\mathbf{Z}_2)$	M5-branes	16
M	$T^2 \times (T^8/\mathbf{Z}_2)$	M2-branes	16
M	$S^1 \times (T^9/\mathbf{Z}_2)$	KK-particles	16
M	$T^6 \times (T^4/\mathbf{Z}_2)$	exceptional S^2	16

Exceptional branes $\Longrightarrow$ Imaginary root α

What is the algebraic connection between α and the $\mathbf{Z}_2$ on E_{10}?

The Answer

The $\mathbf{Z}_2$ action defines a $\mathbf{Z}_2$-gradation of the root lattice of E_{10}.

Every root β has a $\mathbf{Z}_2$-charge

$$Q(\beta) = 0 \text{ or } 1$$

that is determined by the orbifold.

The relation to the
Exceptional branes $\Longrightarrow$ Imaginary root α, is

$$Q(\beta) = (\alpha|\beta) \bmod 2.$$

Example: $T^5/\mathbf{Z}_2$

The orbifold $T^5/\mathbf{Z}_2$:

$$
\begin{array}{rcll}
g_{ij} & \to & g_{ij}, & 1 \le i,j \le 5, \\
g_{iI} & \to & -g_{iI}, & 1 \le i \le 5, \quad 6 \le I \le 10 \\
g_{IJ} & \to & g_{IJ}, & 6 \le I,J \le 10 \\
C_{ijk} & \to & -C_{ijk}, & 1 \le i < j < k \le 5, \\
C_{ijK} & \to & C_{ijK}, & 1 \le i < j \le 5, \quad 6 \le K \le 10 \\
C_{iJK} & \to & -C_{iJK}, & 1 \le i \le 5, \quad 6 \le J < K \le 10 \\
C_{IJK} & \to & C_{IJK}, & 6 \le I < J < K \le 10
\end{array}
$$

[Dasgupta & Mukhi, Witten]

There are 16 exceptional M5-branes in direction $1\ldots5$.

$$\alpha = (2,2,2,2,2,1,1,1,1,1).$$

The Laplacian $\triangle$ on $\exp E_{10}/K_{10}$

$$\int \sqrt{g}\mathcal{R}d^{10}x dt \longrightarrow \int \left\| \frac{d\vec{h}}{d\tilde{\tau}} \right\|^2 d\tilde{\tau}.$$

The metric $\|\cdot\|^2$ has signature $(9,1)$.

Connection with E_{10}:
Space of $\vec{h} \Longrightarrow \widehat{\mathcal{H}}_{\mathbf{R}}$ Cartan subalgebra of E_{10}.

Quantum mechanically we get the Wheeler-deWitt equation: $\mathcal{H}_{0,h}\Psi = 0$.

$$\mathcal{H}_{0,h} = -\sum_{k=1}^{9} \frac{\partial^2}{\partial {h_k}^2} + \frac{1}{9}\left(\sum_{k=1}^{9} \frac{\partial}{\partial h_k}\right)^2$$

Landau levels and brane anti-brane pairs

$$\mathcal{H} = \cdots - e^{2\langle\vec{h},\alpha\rangle}\frac{\partial^2}{\partial\mathcal{C}_\alpha{}^2} - e^{2\langle\vec{h},\beta\rangle}\left(\frac{\partial}{\partial\mathcal{C}_\beta} - \mathcal{C}_\alpha\frac{\partial}{\partial\mathcal{C}_{\gamma,j}}\right)^2 - e^{2\langle\vec{h},\gamma\rangle}\frac{\partial^2}{\partial\mathcal{C}_{\gamma,j}{}^2} + \cdots$$

Set

$$N_{\gamma,j} = -i\frac{\partial}{\partial\mathcal{C}_{\gamma,j}}$$

Then

$$\mathcal{H} = \cdots - e^{2\langle\vec{h},\alpha\rangle}\frac{\partial^2}{\partial\mathcal{C}_\alpha{}^2} - e^{2\langle\vec{h},\beta\rangle}\left(\frac{\partial}{\partial\mathcal{C}_\beta} - N_{\gamma,j}\mathcal{C}_\alpha\right)^2 + \cdots$$

Landau levels:

$$E = (n+\frac{1}{2})\omega, \qquad \omega = 2e^{\langle\vec{h},\alpha+\beta\rangle}|N_{\gamma,j}| = 2e^{\langle\vec{h},\gamma\rangle}|N_{\gamma,j}|.$$

Identify

$$n = \#\text{pairs}$$

But we need

$$N_{\gamma,j} \in 2\pi\mathbf{Z},$$

which is strange!

Summary

- Prime isotropic imaginary roots correspond to Minkowski branes.

- The exceptional branes in Z_2 orbifolds correspond to the imaginary root that defines the Z_2 charge.

- We have a "K-theory like" characterization:

$$\begin{array}{ccc} \text{Real roots} & \Longleftrightarrow & \text{Fluxes} \\ \text{Prime isotropic imaginary roots} & \Longleftrightarrow & \text{Charges} \end{array}$$

- Energy levels with ΔE proportional to brane masses arise as Landau-levels of QM on a coset of E_{10}. (But a prefactor of 2π requires a wrong periodicity for the variables!)

- Interactions of branes with branes and of branes with fluxes can be characterized by the inner product of the corresponding roots.