# From S-Duality To Chern-Simons Theory via Minimal-length Strings

**Ori Ganor**

**UC Berkeley and LBNL**

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Dualities in Physics and Mathematics

Kavli Institute for Theoretical Physics, UCSB

# Based on

- Yoon Pyo Hong and OG, "S-duality and Chern-Simons Theory," [arXiv:hep-th/0812.1213]
- Yoon Pyo Hong and OG, "From S-Daulity to Chern-Simons Theory via Minimal-length Strings," [arXiv:hep-th/0904.????]

# S-duality

$$\tau \equiv \frac{4\pi i}{g^2_{\mathrm{YM}}} + \frac{\theta}{2\pi}$$

$$\mathbf{s} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$$

$$\tau \to \frac{\mathbf{a}\tau + \mathbf{b}}{\mathbf{c}\tau + \mathbf{d}}\,.$$

# S-duality's action on states $[G = U(1)]$

$$\text{Temporal gauge:} \qquad A_0 = 0.$$

$$\widetilde{\Psi}(A) \equiv \int [\mathcal{D}\widetilde{A}]\mathcal{S}(A,\widetilde{A})\Psi(\widetilde{A})$$

$$\tau \to \frac{\mathbf{a}\tau + \mathbf{b}}{\mathbf{c}\tau + \mathbf{d}}\,, \qquad E_i \to \mathbf{a}E_i + \mathbf{b}B_i\,, \quad B_i \to \mathbf{c}E_i + \mathbf{d}B_i\,.$$

[Lozano; Gaiotto & Witten]

$$\mathcal{S}(A,\widetilde{A}) = \exp\left\{\frac{i}{4\pi\mathbf{c}}\int(\mathbf{d}A \wedge dA - 2\widetilde{A}\wedge dA + \mathbf{a}\widetilde{A}\wedge d\widetilde{A})\right\}.$$

$$\widetilde{E}_i\mathcal{S} = \mathcal{S}(\mathbf{a}E_i + \mathbf{b}B_i), \qquad \widetilde{B}_i\mathcal{S} = \mathcal{S}(\mathbf{c}E_i + \mathbf{d}B_i).$$

$$E_i \equiv -2\pi i\delta/\delta A_i$$

# $U(1)$ Chern-Simons from S-duality

$$\widetilde{\Psi}\{A\} \equiv \int [\mathcal{D}\widetilde{A}]\mathcal{S}(A,\widetilde{A})\Psi(\widetilde{A})$$

$$\mathcal{S}(A,\widetilde{A}) = \exp\left\{\frac{i}{4\pi\mathbf{c}}\int(\mathbf{d}A\wedge dA - 2\widetilde{A}\wedge dA + \mathbf{a}\widetilde{A}\wedge d\widetilde{A})\right\}.$$

$$A = \widetilde{A} \Longrightarrow \mathcal{I}(A) \equiv \frac{\mathbf{a}+\mathbf{d}-2}{4\pi\mathbf{c}}\int A\wedge dA\,.$$

$$\text{CS level:} \qquad k \equiv \left(\frac{\mathbf{a}+\mathbf{d}-2}{\mathbf{c}}\right).$$

What is the generalization for $G = U(n)$?

# Selfduality

$$\tau = \frac{\mathbf{a}\tau + \mathbf{b}}{\mathbf{c}\tau + \mathbf{d}} \Longrightarrow \mathbf{c}\tau + \mathbf{d} = e^{i\upsilon}.$$

At a selfdual $\tau$ we can compactify on a circle with an S-twist.

# Selfduality

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What is the low-energy limit?

Topological for $n < 4$ (up to gravitational CS term).

$$k \equiv \left(\frac{\mathbf{a} + \mathbf{d} - 2}{\mathbf{c}}\right) = \text{integer}.$$

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In this talk:

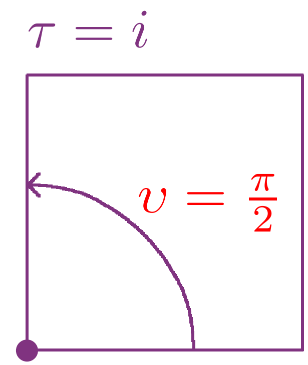
1. Identify the Hilbert space $\mathcal{H}$ of states on $T^2$ and calculate its dimension.

2. Identify the $\mathrm{SL}(2, \mathbb{Z})$ (large $T^2$ automorphisms) action on $\mathcal{H}$.

# The self-dual $\tau$'s

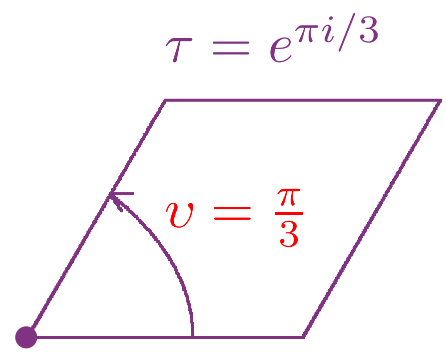
$$\mathbf{s} \equiv \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\tau \longrightarrow -\frac{1}{\tau} \qquad |k| = \left|\frac{\mathbf{a}+\mathbf{d}-2}{\mathbf{c}}\right| = 2$$



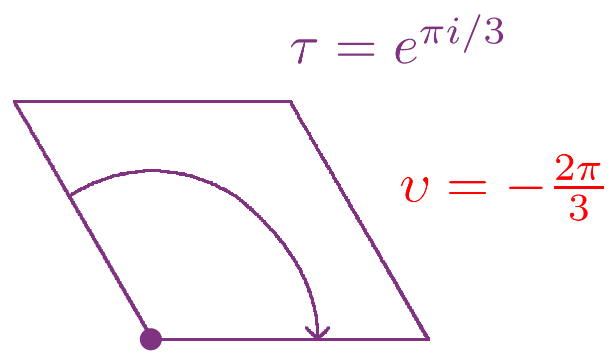

$$\mathbf{s} \equiv \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

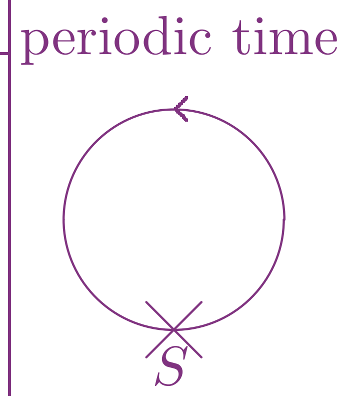
$$\tau \longrightarrow \frac{\tau-1}{\tau} \qquad |k| = \left|\frac{\mathbf{a}+\mathbf{d}-2}{\mathbf{c}}\right| = 1$$




$$\mathbf{s} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\tau \longrightarrow \frac{\tau-1}{\tau} \qquad |k| = 3$$



# $N = 4$ **Super Yang-Mills**

| | | |
|---|---|---|
| $A_\mu$ | gauge field | $\mu = 0 \ldots 3$ |
| $\Phi^I$ | adjoint-valued scalars | $I = 1 \ldots 6$ |
| $\psi^a_\alpha$ | adjoint-valued spinors | $a = 1 \ldots 4$ and $\alpha = 1, 2$ |
| $\overline{\psi}_{a\dot{\alpha}}$ | complex conjugate spinors | $a = 1 \ldots 4$ and $\dot{\alpha} = \dot{1}, \dot{2}$ |
| $Q_{a\alpha}$ | SUSY generators | $a = 1 \ldots 4$ and $\alpha = 1, 2$ |
| $\overline{Q}^a_{\dot{\alpha}}$ | complex conjugate generators | $a = 1 \ldots 4$ and $\dot{\alpha} = \dot{1}, \dot{2}$ |

$$Z^1 = \Phi^1 + i\Phi^4, \qquad Z^2 = \Phi^2 + i\Phi^5, \qquad Z^3 = \Phi^3 + i\Phi^6.$$

# Supersymmetry

$$\mathbf{s}: \tau \rightarrow \frac{\mathbf{a}\tau + \mathbf{b}}{\mathbf{c}\tau + \mathbf{d}}$$

$$\mathbf{s}: Q_{a\alpha} \rightarrow \left( \frac{\mathbf{c}\tau + \mathbf{d}}{|\mathbf{c}\tau + \mathbf{d}|} \right)^{1/2} Q_{a\alpha} = e^{\frac{i\upsilon}{2}} Q_{a\alpha}$$

[Kapustin & Witten]

$$\mathbf{s} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Longrightarrow \upsilon = \frac{\pi}{2}$$

## R-Symmetry

$Spin(6) \simeq SU(4)$

$$\gamma \equiv \begin{pmatrix} e^{i\varphi_1} & & & \\ & e^{i\varphi_2} & & \\ & & e^{i\varphi_3} & \\ & & & e^{i\varphi_4} \end{pmatrix} \in SU(4)\,, \qquad \left(\sum_a \varphi_a = 0\right)\,,$$

acts as

$$\gamma(\psi^a_\alpha) = e^{i\varphi_a}\,\psi^a_\alpha\,, \qquad \gamma(\overline{\psi}_{a\alpha}) = e^{-i\varphi_a}\,\overline{\psi}_{a\alpha}\,, \qquad a = 1\ldots 4.$$

$$\gamma(Z^k) = e^{i(\varphi_k+\varphi_4)}\,Z^k\,, \qquad k = 1\ldots 3.$$

# Combined R-S- action

$$Q_{a\alpha} \to e^{\frac{i\upsilon}{2} - i\varphi_a} Q_{a\alpha} \,.$$

$\Longrightarrow N = 2r$ invariant generators

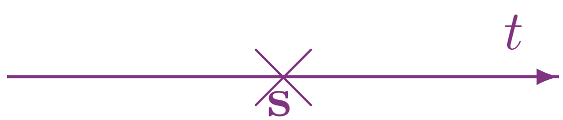
$r = \#\{a \text{ for which } e^{i\varphi_a} = e^{i\upsilon/2}\}$

# R- and S- twisted boundary conditions


$$\Phi(x = 0^-) = \gamma[\Phi(x = 0^+)]$$

$$Z^k(x = 0^-) = e^{i(\varphi_k+\varphi_4)}\Phi(x = 0^+), \qquad k = 1, 2, 3$$

$\ldots$



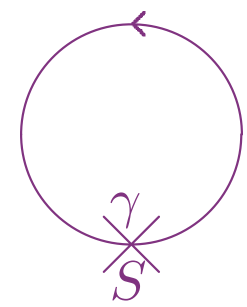

$$\Psi(A, \ldots)|_{t=0^+} = \int [\mathcal{D}\widetilde{A}]\mathcal{S}(A, \widetilde{A})\Psi(\widetilde{A}, \ldots)|_{t=0^-}$$

# SUSY in 2+1D

$$\Longrightarrow N = 2r, \qquad r = \#\{a \text{ for which } e^{i\varphi_a} = e^{i\upsilon/2}\}$$

$$\gamma = \begin{pmatrix} e^{\frac{i}{2}\upsilon} & & & \\ & e^{\frac{i}{2}\upsilon} & & \\ & & e^{\frac{i}{2}\upsilon} & \\ & & & e^{-\frac{3i}{2}\upsilon} \end{pmatrix} \Longrightarrow N = 6$$

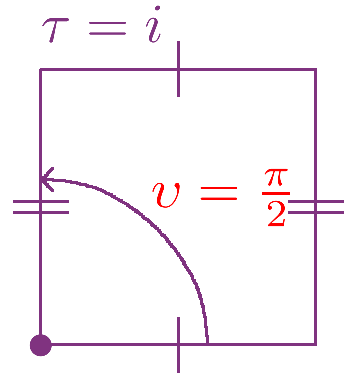
$$\gamma = \begin{pmatrix} e^{\frac{i}{2}\upsilon} & & & \\ & e^{\frac{i}{2}\upsilon} & & \\ & & e^{-i(\upsilon+\varphi_4)} & \\ & & & e^{i\varphi_4} \end{pmatrix} \Longrightarrow N = 4$$




$\gamma =$ R-symmetry twist $\qquad e^{i\upsilon} \equiv \mathbf{c}\tau + \mathbf{d}$

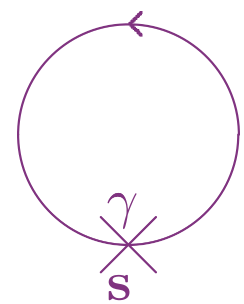
$$\mathbf{s} \equiv \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\tau \longrightarrow -\frac{1}{\tau}$$




$\mathbf{s}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$
$\gamma(v)$

$N = 4$ SYM




$N = 6$
in 2+1D

IR???

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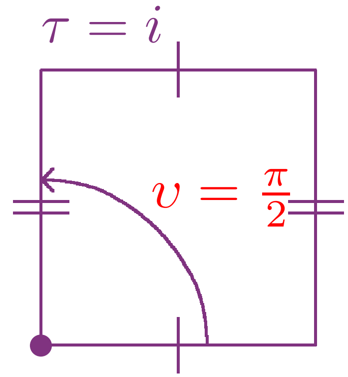
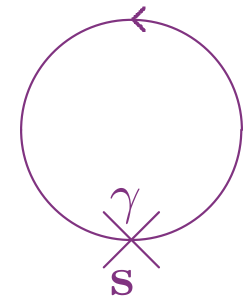



$\mathbf{s}(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})$
$\gamma(\upsilon)$

$N = 4$ SYM

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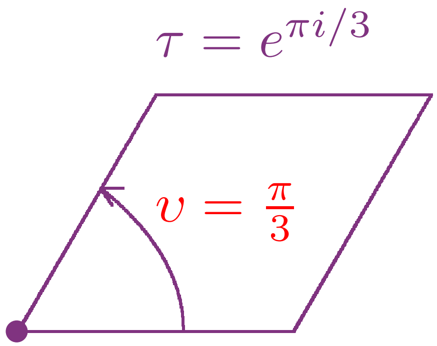


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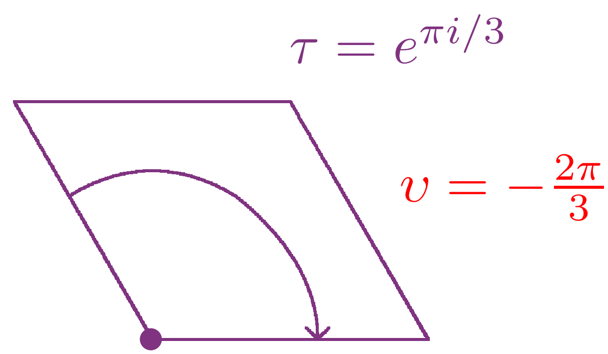


$N = 6$
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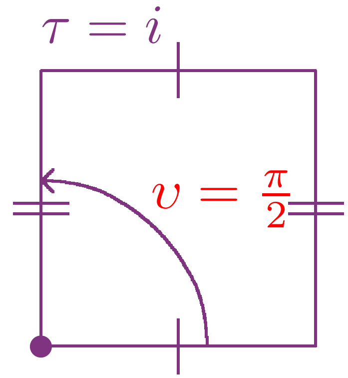
$\boxed{\text{CS at } k = 2?}$

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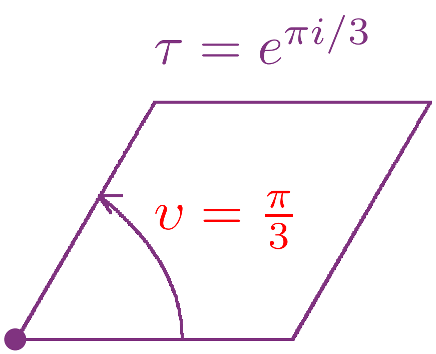
$$\tau \longrightarrow \frac{\tau - 1}{\tau}$$

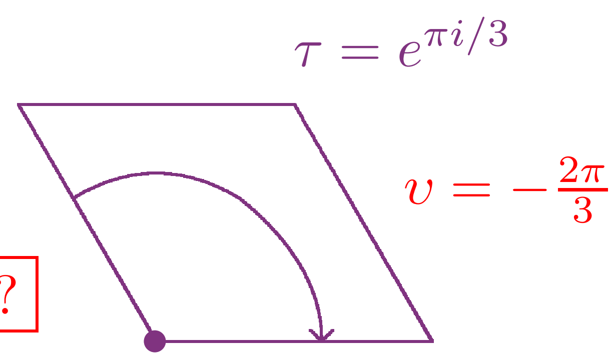
$\boxed{\text{CS at } k = 1?}$




$$\mathbf{s} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

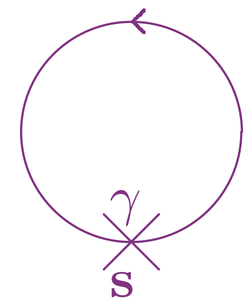
$$\tau \longrightarrow \frac{\tau - 1}{\tau}$$

$\boxed{\text{CS at } k = 3?}$




$\mathbf{s}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$
$\gamma(\upsilon)$

$N = 4$ SYM




$N = 6$
in 2+1D

IR???

# Is there a Coulomb branch?

$$Z^j(x_3 + 2\pi R) = e^{i\upsilon} Z^j(x_3)\,, \qquad j = 1,2,3 \quad \text{(the complex scalars)}$$

- for $\tau = i$ and $\mathbf{s} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, $\upsilon = \frac{\pi}{2}$;

- for $\tau = e^{\pi i/3}$ and $\mathbf{s} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, $\upsilon = \frac{\pi}{3}$;

- for $\tau = e^{\pi i/3}$ and $\mathbf{s} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$, $\upsilon = -\frac{2\pi}{3}$;

$$\langle Z^j \rangle = e^{i\upsilon} \langle Z^j(x_3) \rangle \Longrightarrow \langle Z^j \rangle = 0.$$

# Is there a Coulomb branch...?

$$\Lambda^{-1} Z^j(x_3 + 2\pi R)\Lambda = e^{i\upsilon} Z^j(x_3)\,, \qquad \Lambda \in \text{Weyl group} = S_n \subset U(n)$$

- for $\tau = i$ and $\mathbf{s} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, $\upsilon = \frac{\pi}{2}$;
- for $\tau = e^{\pi i/3}$ and $\mathbf{s} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, $\upsilon = \frac{\pi}{3}$;
- for $\tau = e^{\pi i/3}$ and $\mathbf{s} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$, $\upsilon = -\frac{2\pi}{3}$;

$$n < \frac{2\pi}{|\upsilon|} \Longrightarrow \langle Z^j \rangle = 0 \Longrightarrow \text{No Coulomb branch!}$$

# STRING THEORY

# String Theory

($N = 4$ SYM at selfdual $\tau$) on

$$S^1 \,(\text{direction } 3) \times M_3 \,(\text{directions } 0,1,2).$$

$$M_3 \to T^2 \,(\text{directions } 1,2) \times \mathbb{R} \,(\text{directions } 0)$$

# String Theory

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$$M_3 \longrightarrow T^2 \,(\text{directions } 1, 2) \times \mathbb{R} \,(\text{directions } 0)$$

$$\text{D3-branes in type-IIB } (\tau) \xrightarrow{\text{U-duality}} \text{Make S-duality geometrical.}$$

We'll have another torus $T^2(\tau)$ instead of $T^2$.

# U-duality

| type | brane | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| IIB | D3 | $=$ | $=$ | $\div$ | | | | | | | $\times$ | T on 1: |
| IIA | D2 | $\circ$ | $=$ | $\div$ | | | | | | | $\times$ | to M: |
| M | M2 | $\circ$ | $=$ | $\div$ | | | | | | | $\circ$ | on 2: |
| IIA | F1 | $\circ$ | $\times$ | $\div$ | | | | | | | $\circ$ | |

Legend:

- $\times$ direction doesn't exist in the theory;
- $=$ a direction that the brane wraps;
- $\div$ a direction that the brane wraps and has the S-R-twist;
- $\circ$ a compact direction that the brane doesn't wrap;

# Counting fixed-points

| type | brane | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| IIB | D3 | $=$ | $=$ | $\div$ | | | | | | | $\times$ |
| IIA | F1 | $\circ$ | $\times$ | $\div$ | | | | | | | $\circ$ |

$$\tau = i \Longrightarrow g_{\mathrm{IIB}} = 1 \Longrightarrow R_1 = R_{10}.$$

Directions $1, 10$ form a $T^2$ of complex structure $\tau$;

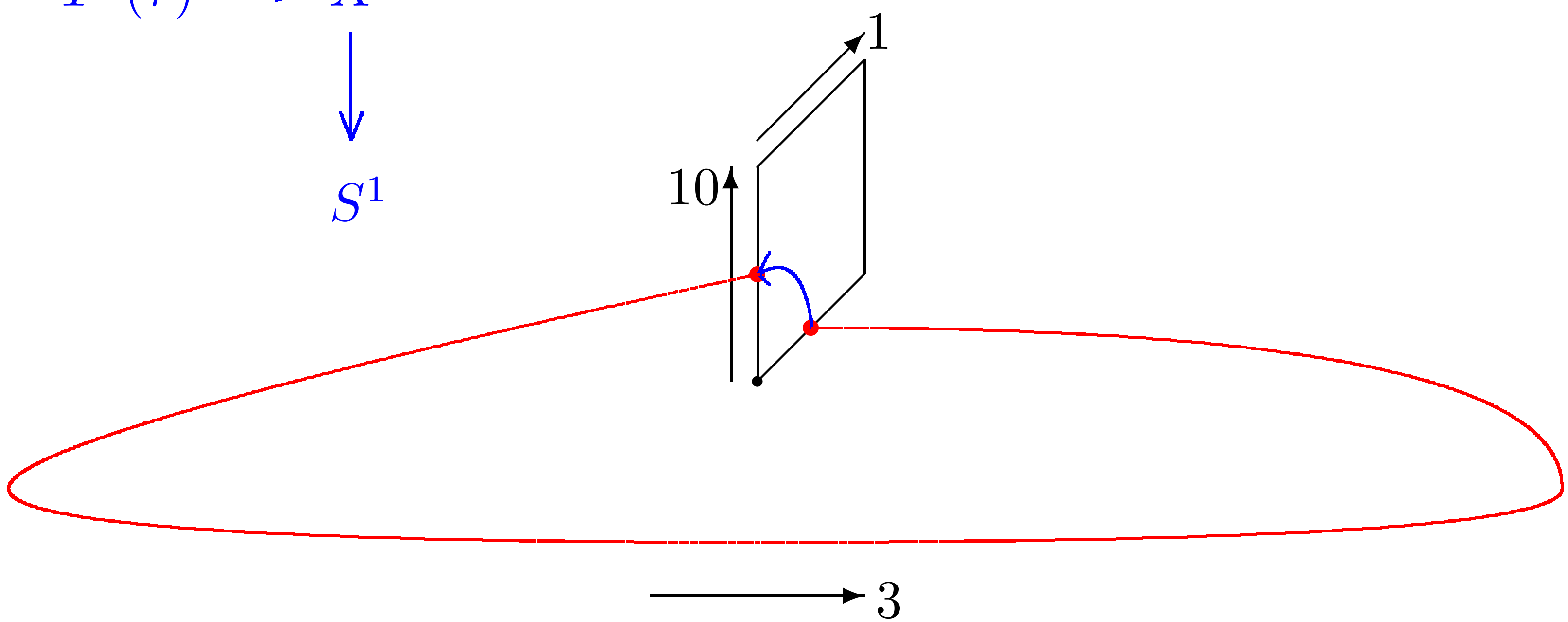
F1-strings are $n$ points in directions $1, 10$;

F1-strings are wound in direction $3$;

$T^2$ (directions $1, 10$) fibered over $S^1$ (direction 3):

Geometrical twist and wound string

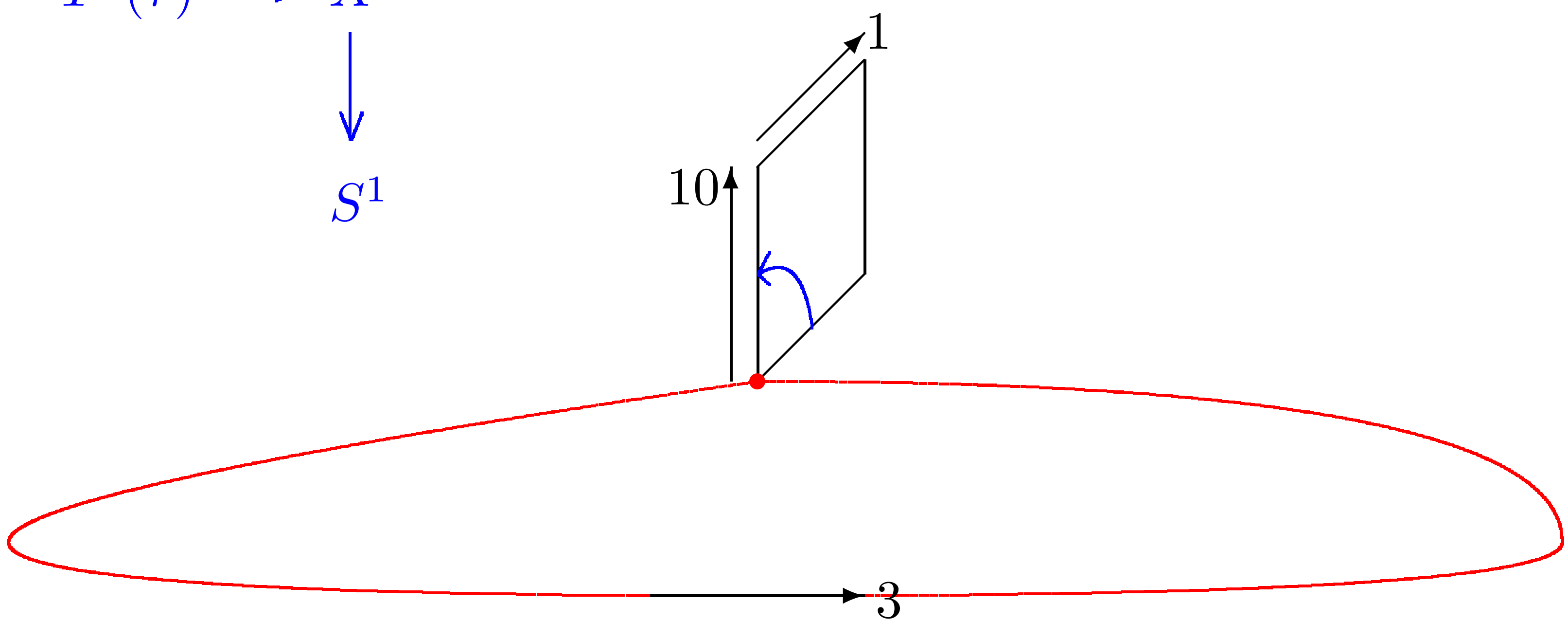
$$\begin{array}{ccc} T^2(\tau) & \longrightarrow & X \\ & & \downarrow \\ & & S^1 \end{array}$$



$T^2$ (directions $1, 10$) fibered over $S^1$ (direction 3):

Geometrical twist and wound string
Minimal energy (length) configuration: find fixed points of twist!

$$\begin{tikzcd} T^2(\tau) \arrow[r] & X \arrow[d] \\ & S^1 \end{tikzcd}$$



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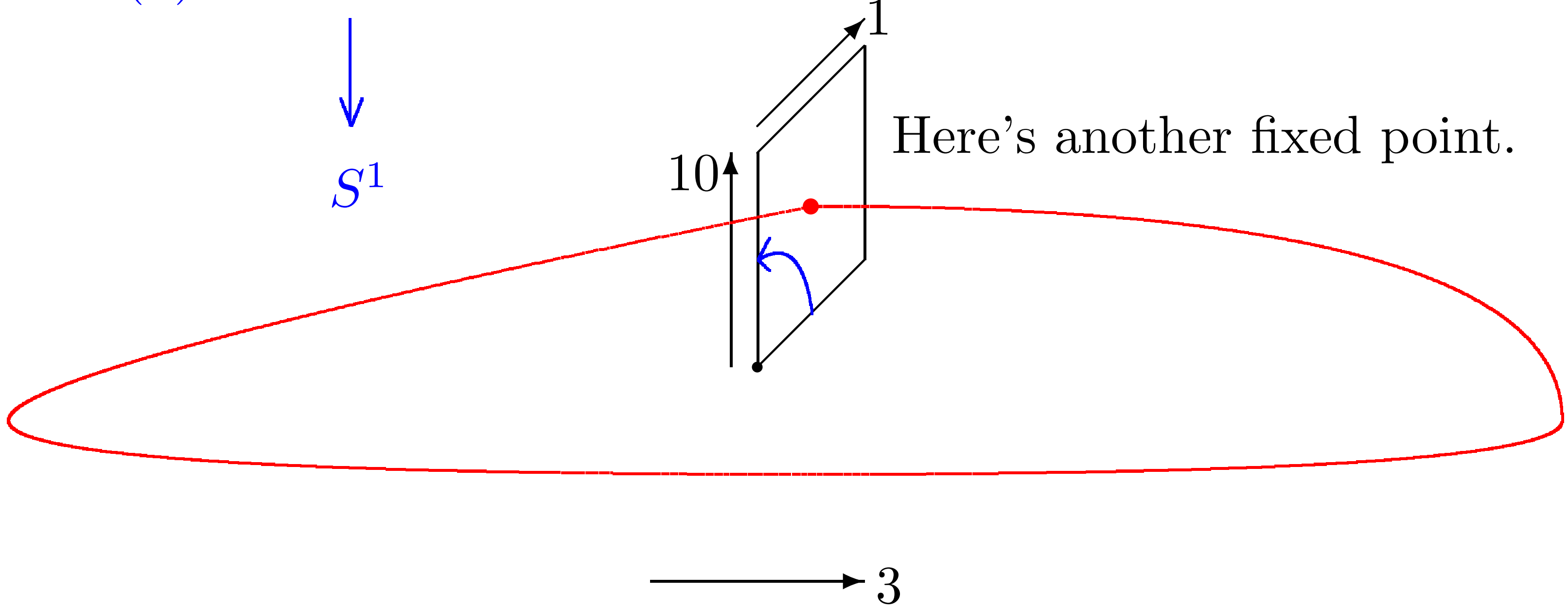
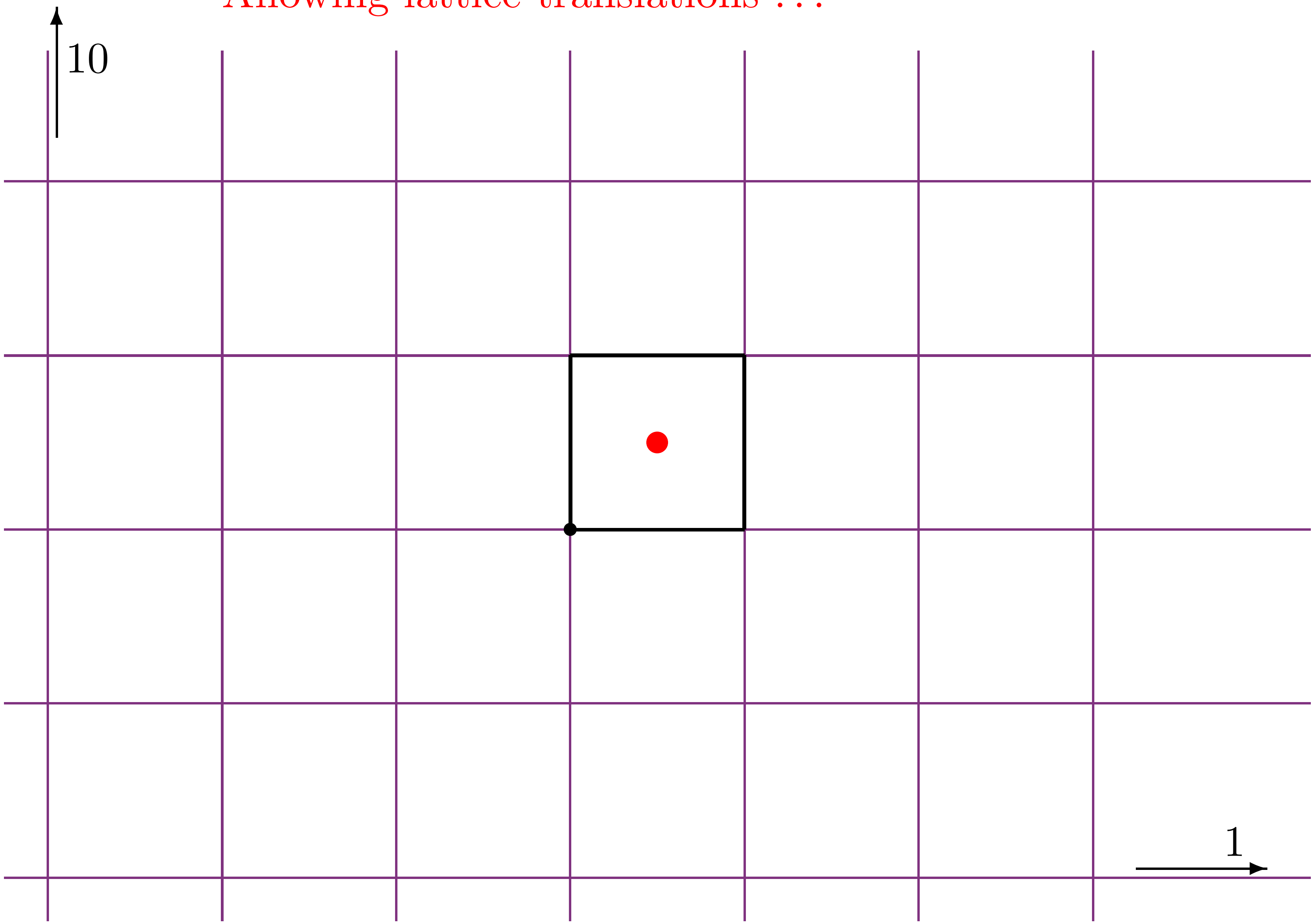


Allowing lattice translations . . .



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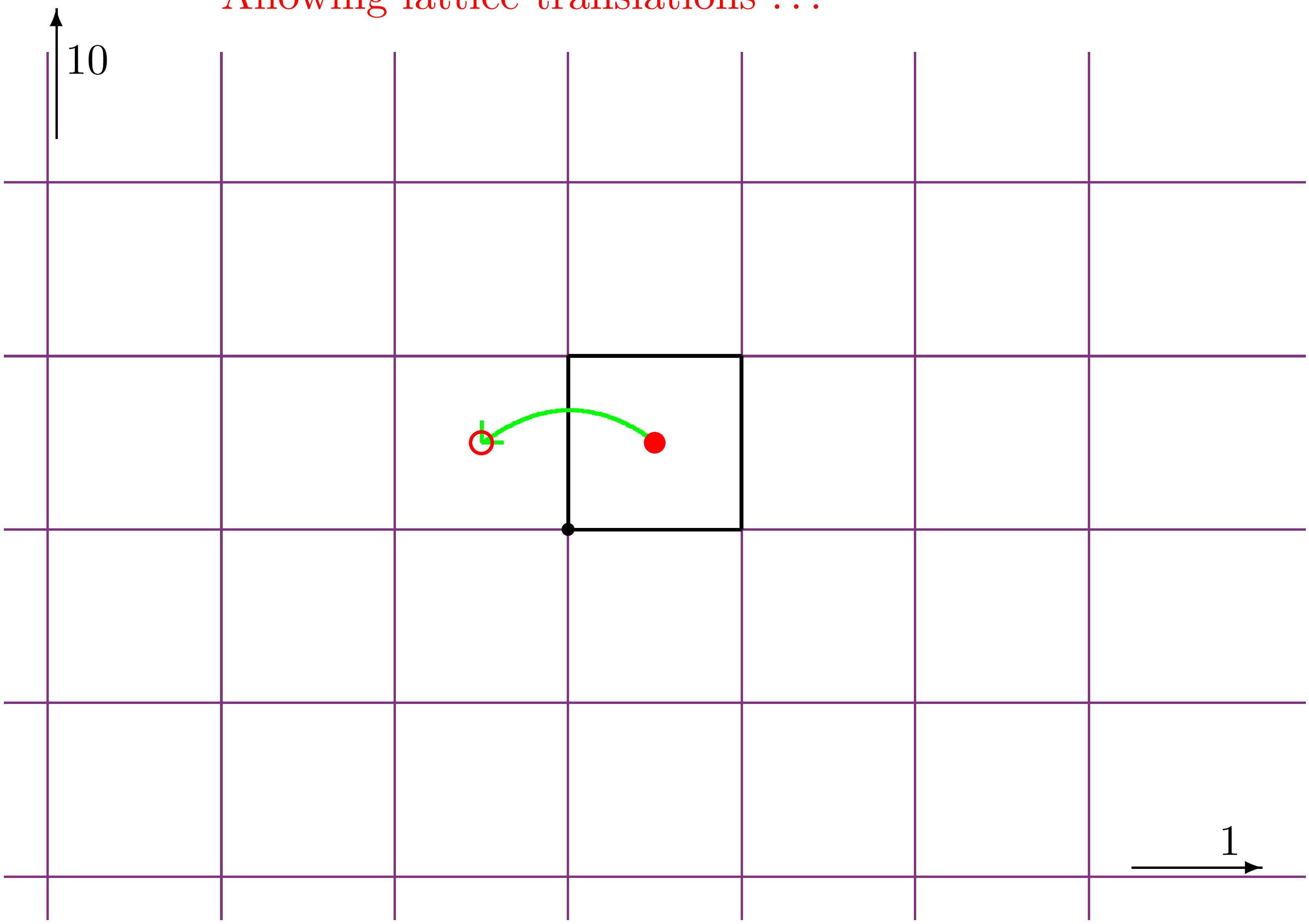
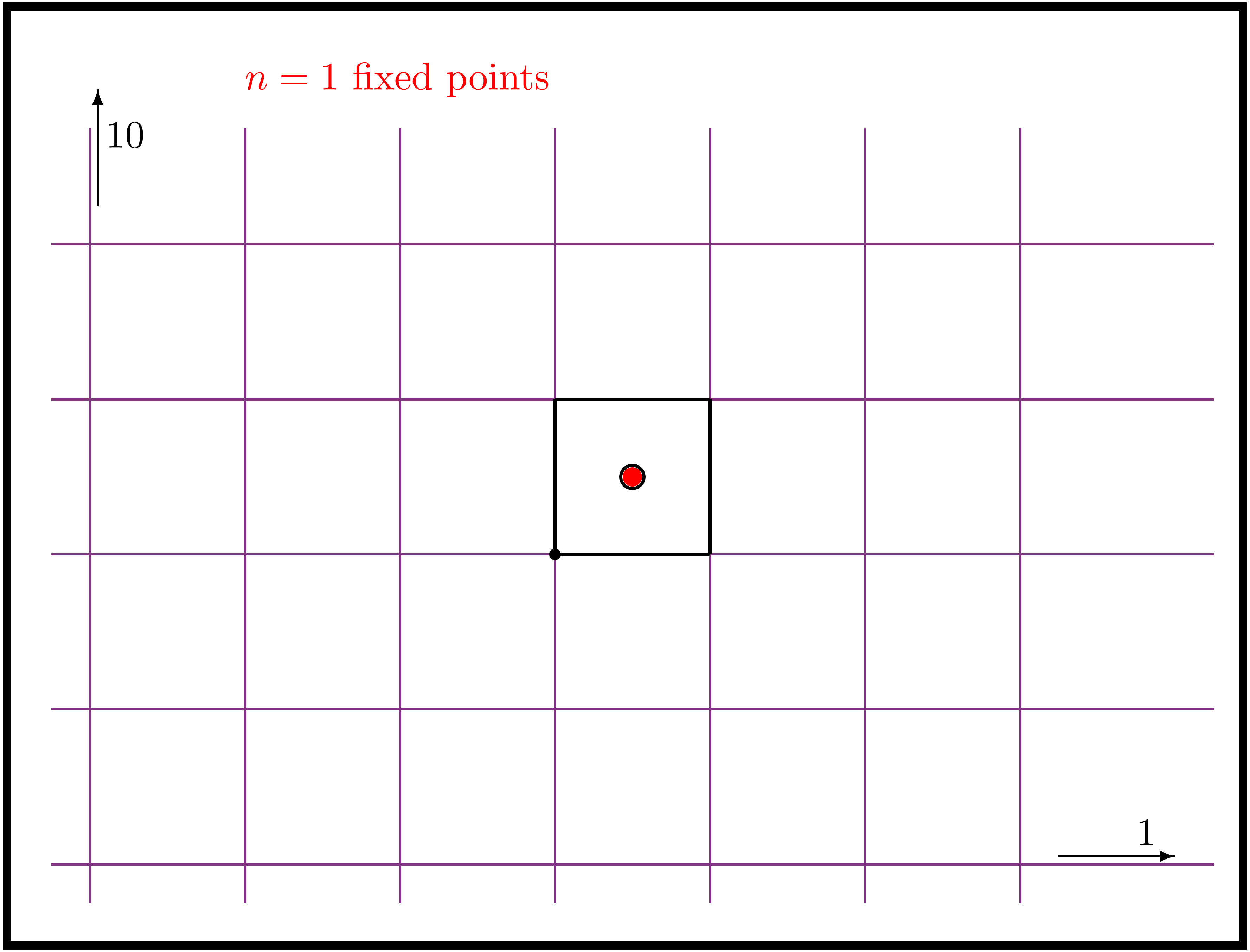
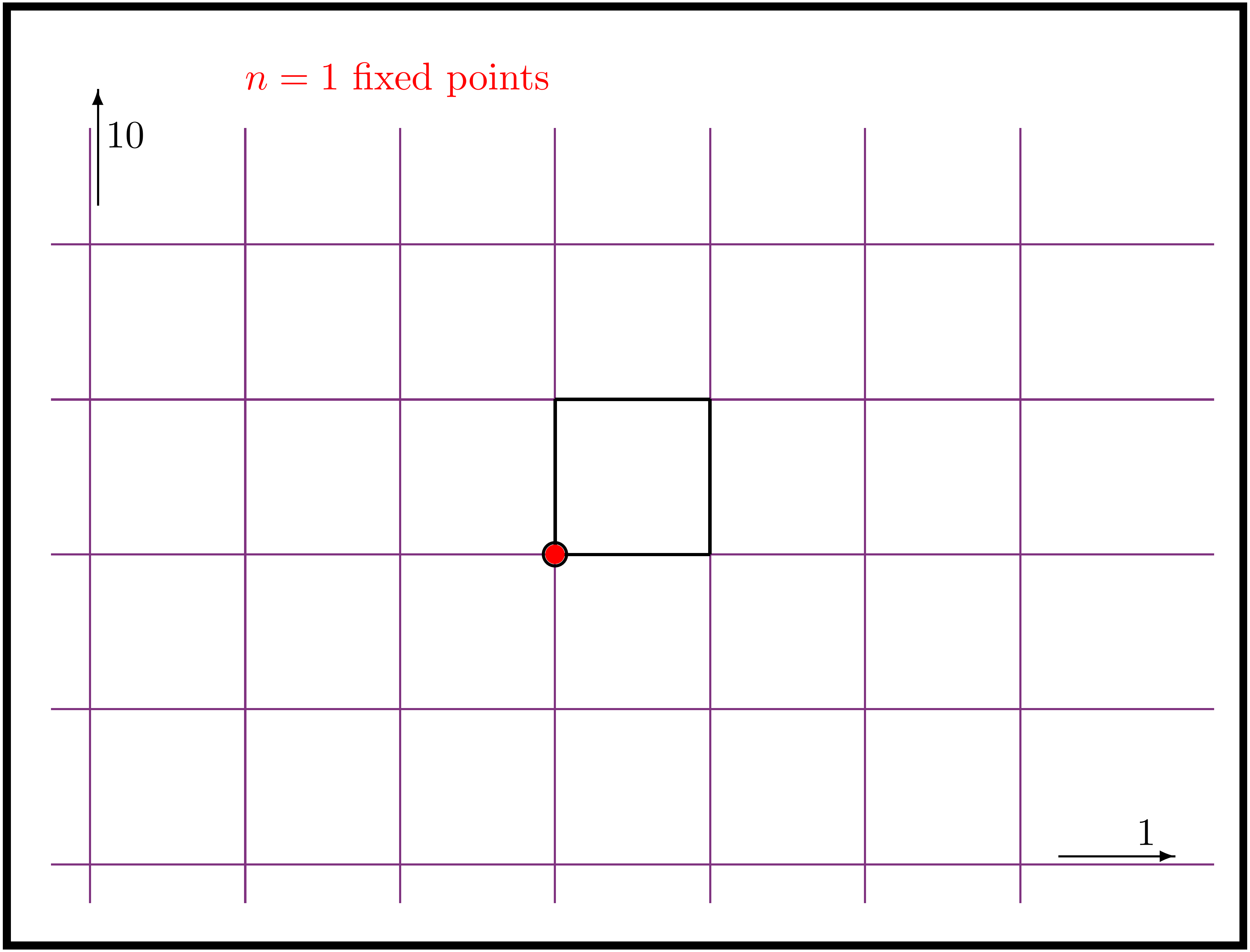
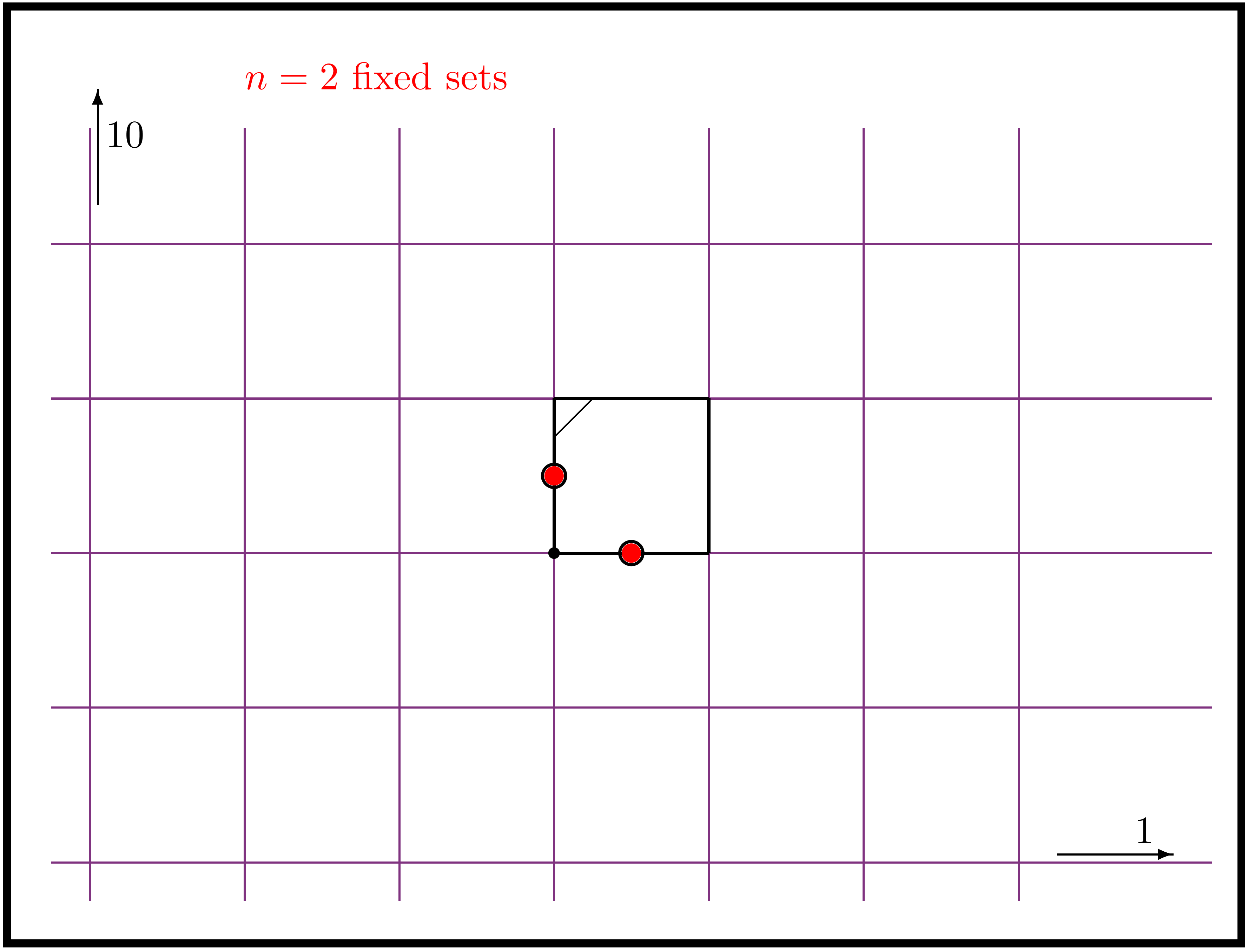
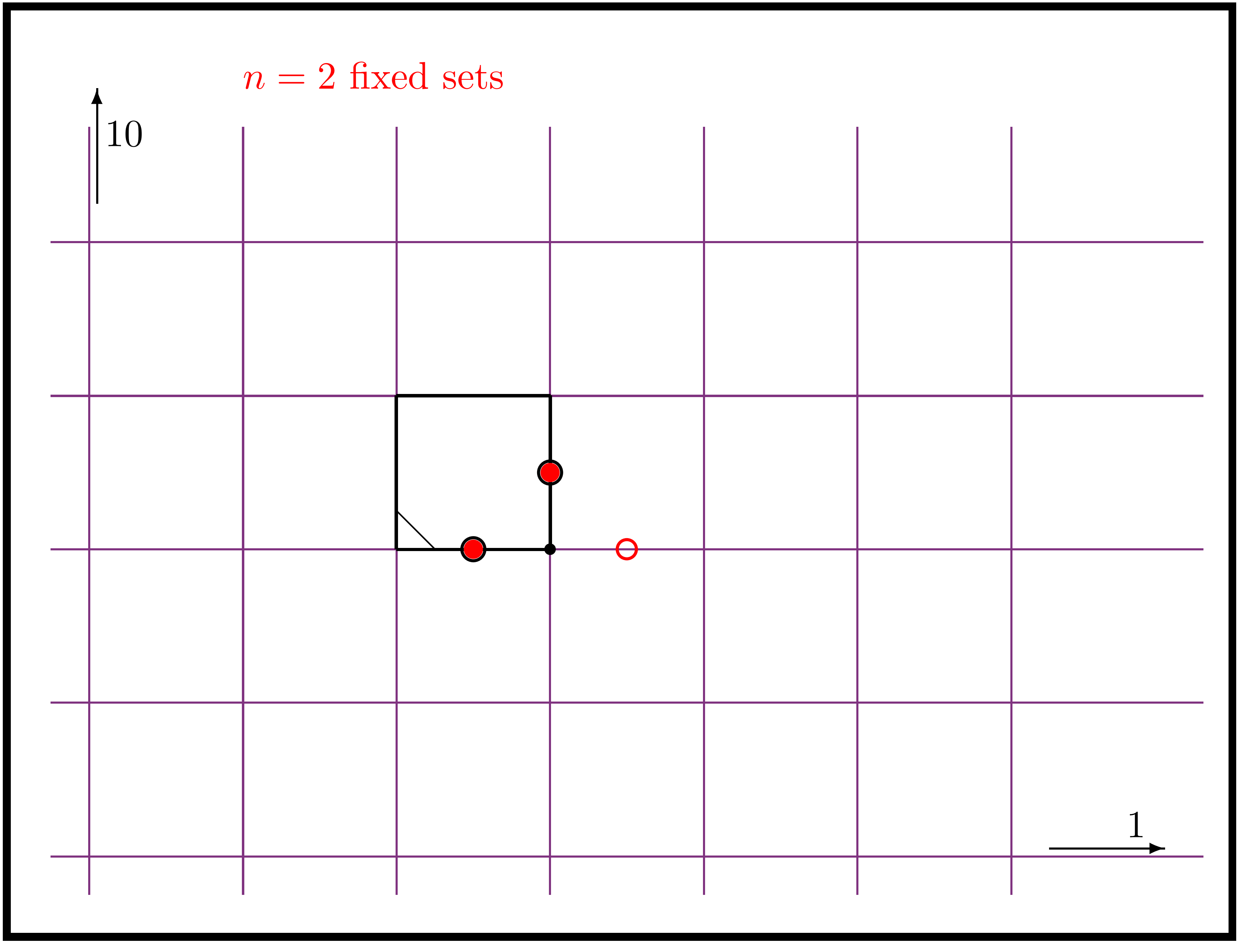
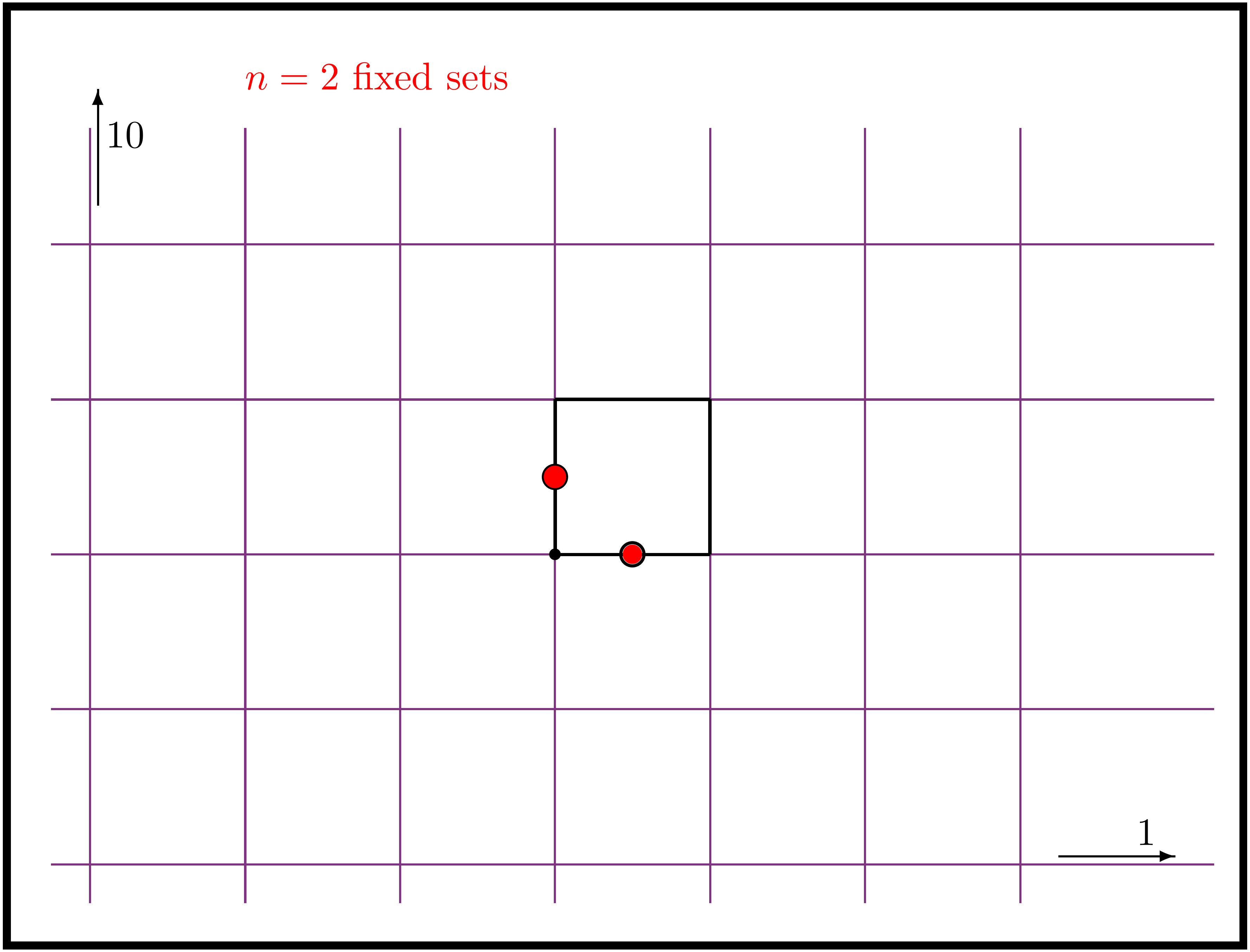


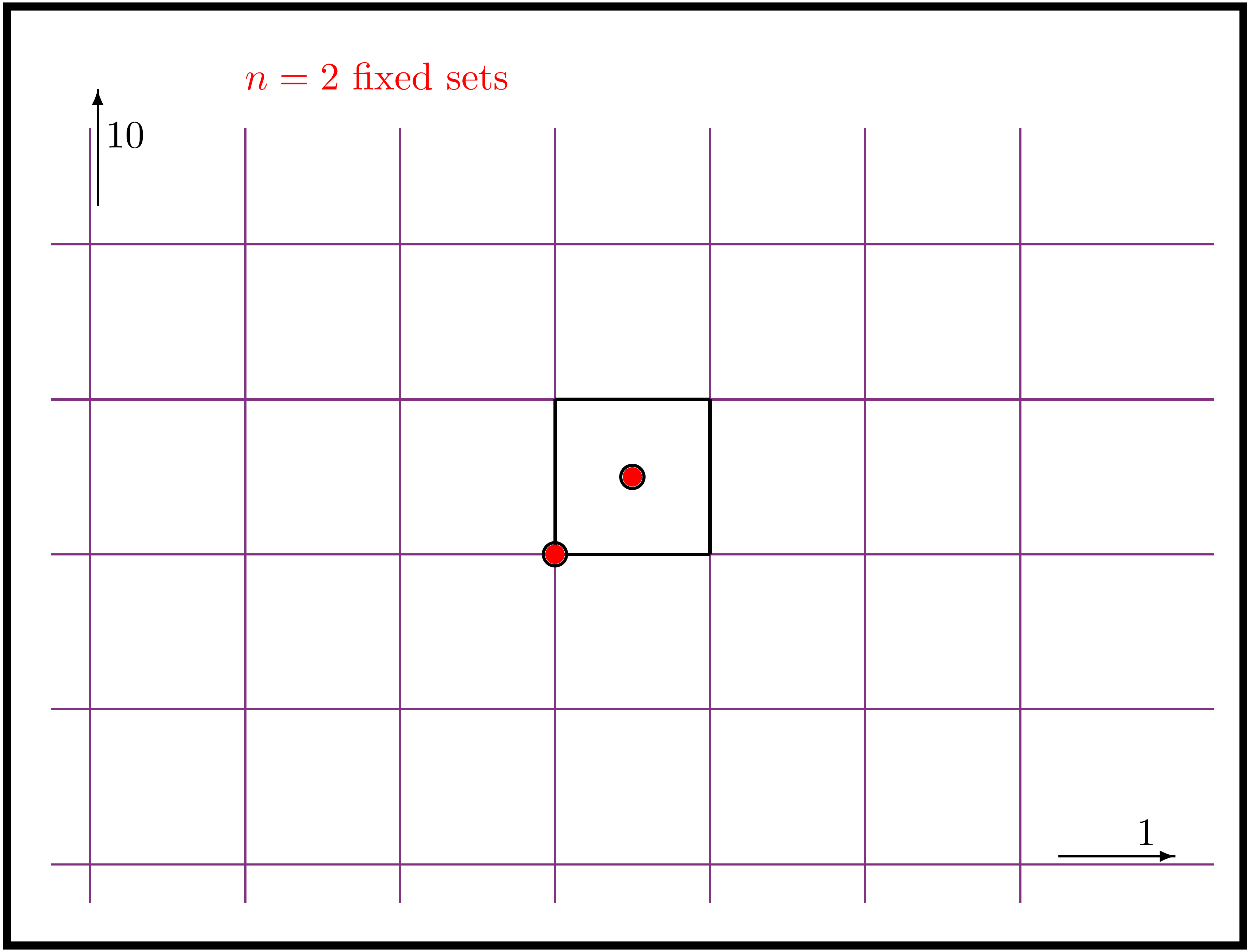

$n = 1$ fixed points
10
1


$n = 1$ fixed points
10
1

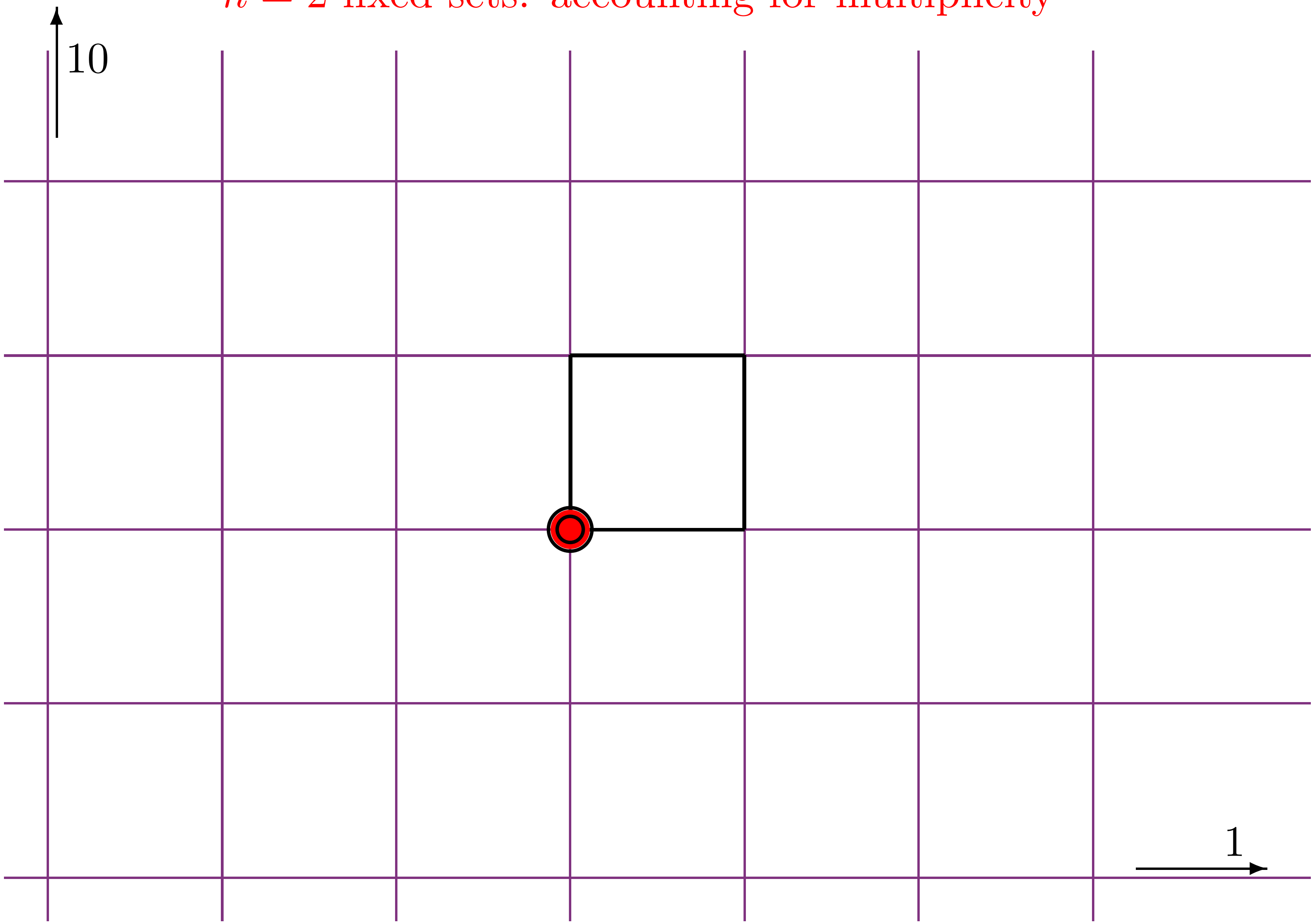

$n = 2$ fixed sets
10
1


$n = 2$ fixed sets
10
1


$n = 2$ fixed sets
10
1

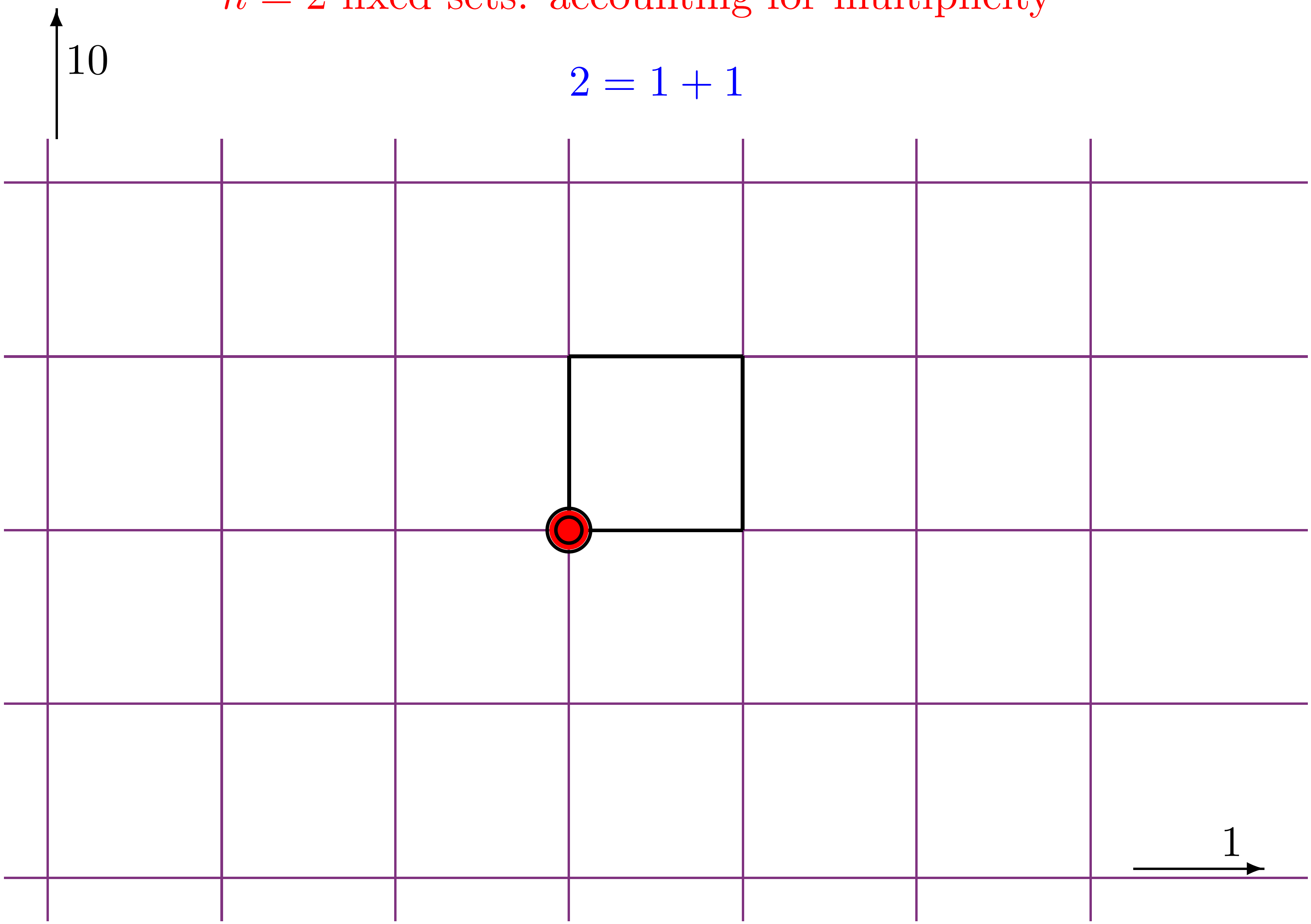

$n = 2$ fixed sets
10
1

## $n = 2$ fixed sets: accounting for multiplicity



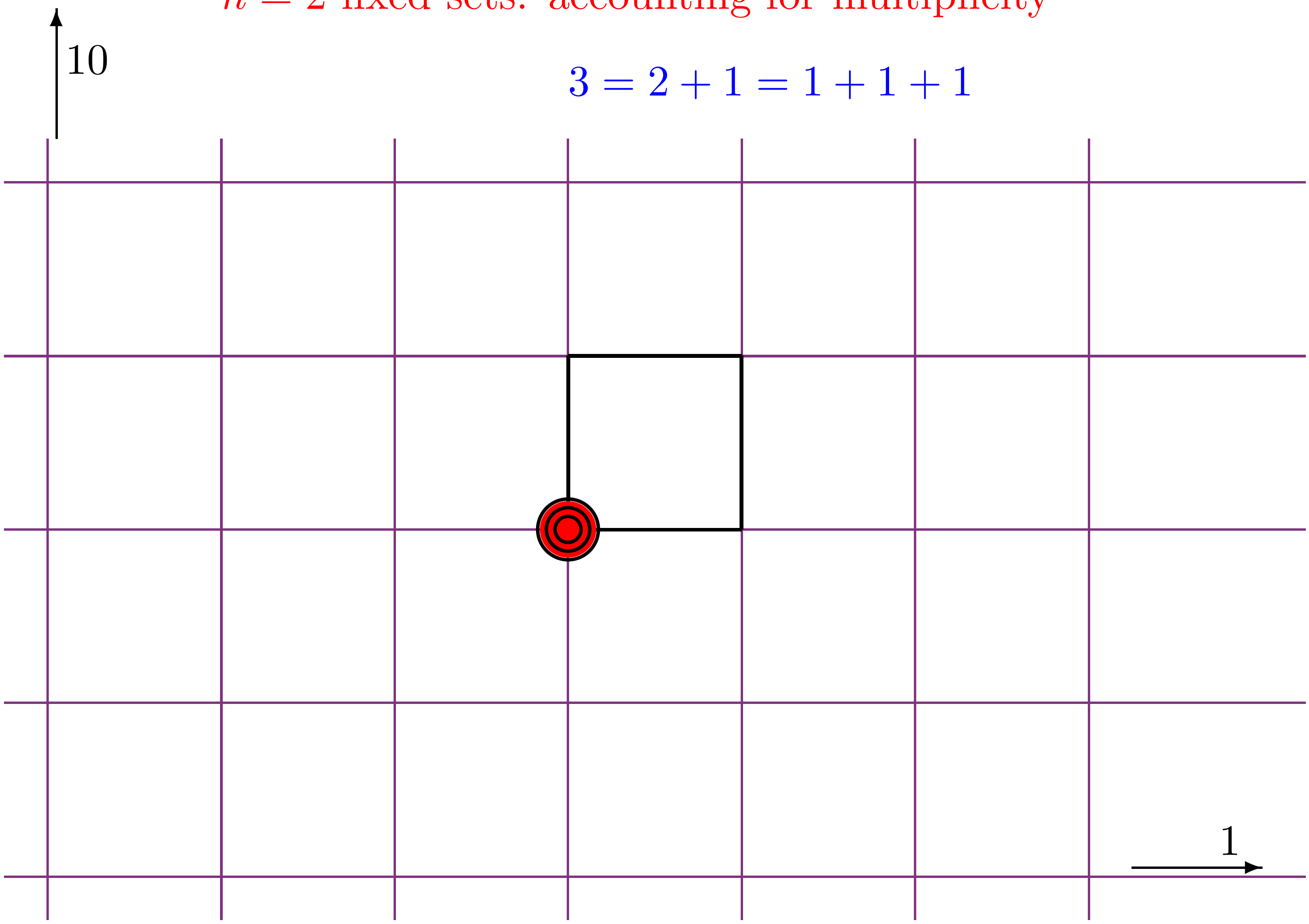
$n = 2$ fixed sets: accounting for multiplicity

$2 = 1 + 1$



$n = 2$ fixed sets: accounting for multiplicity

$3 = 2 + 1 = 1 + 1 + 1$



# Counting fixed-points . . .

Directions $1, 10$ form a $T^2$ of complex structure $\tau$;

F1-strings are $n$ points in directions $1, 10$;

F1-strings are wound in direction $3$;

S-R-twist is entirely geometrical!

It is a rotation by $\upsilon = \pi/2$ of $T^2$;

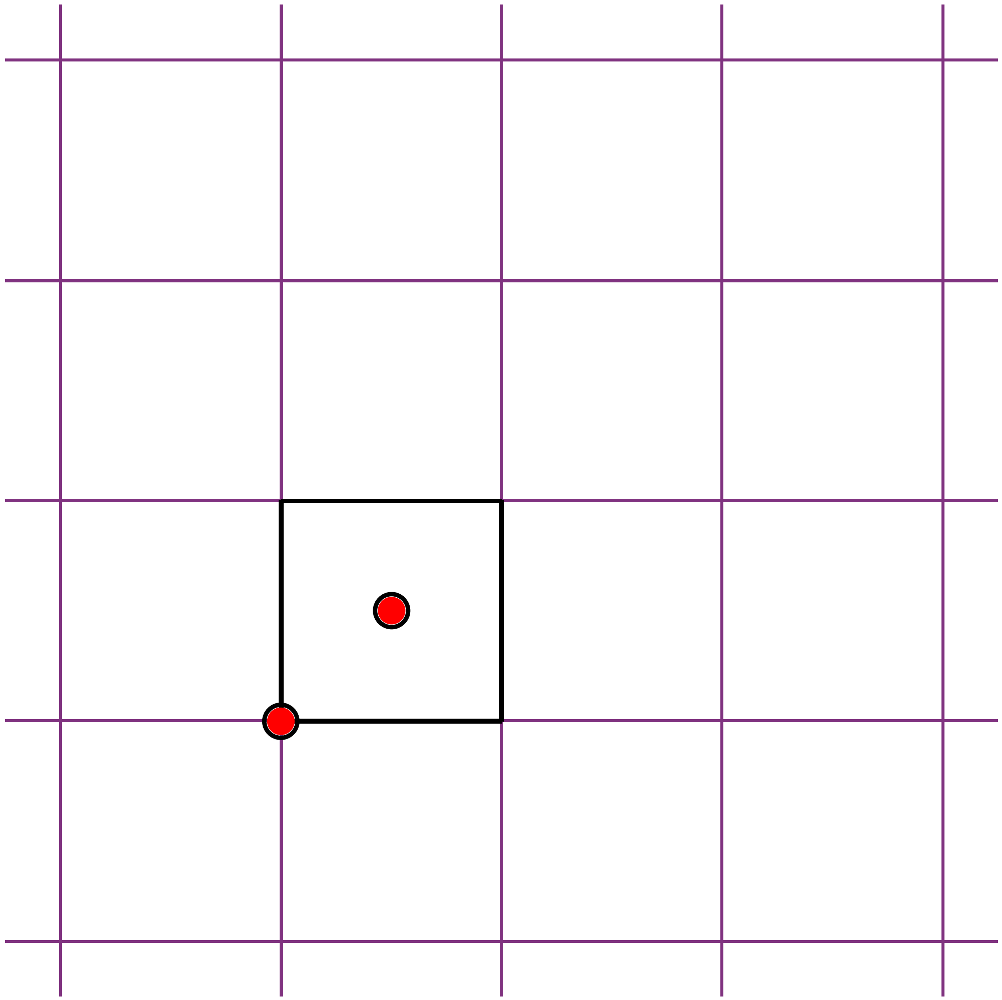
Need to find fixed points of this rotation (up to $S_n$);

$\{z_{\sigma(1)}, \ldots, z_{\sigma(n)}\} = \{z_1, \ldots, z_n\}$ up to $\mathbb{Z} + \mathbb{Z}\tau$;

One Ramond-Ramond ground state for each fixed point.

## Number of states for $U(1)$ $\tau = i$ $(k = 2)$

(rotation by $\upsilon = \pi/2$)



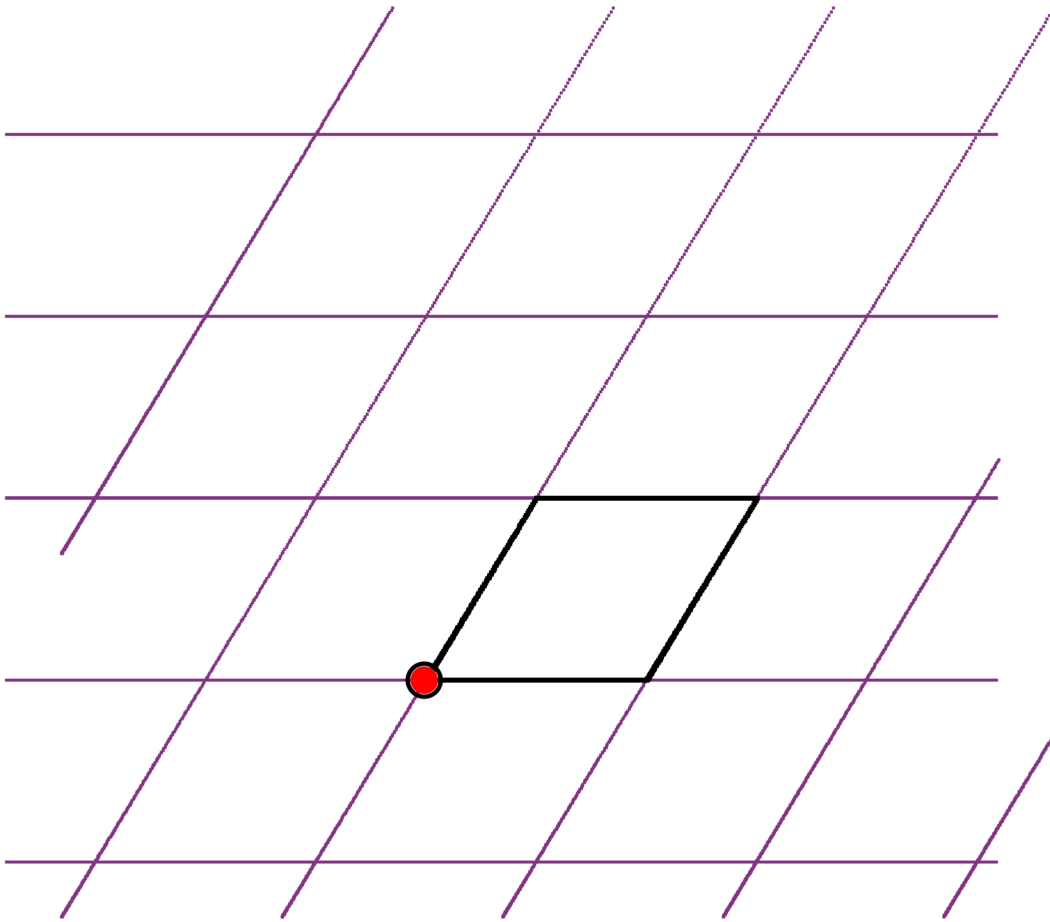
$U(1)$ CS on $T^2$ at level $k$

has $k$ independent states

$\tau = i \Rightarrow 2$ states.

## Number of states for $U(1)$ $\tau = i\pi/3$ $(k=1)$

(rotation by $\upsilon = \pi/3$)



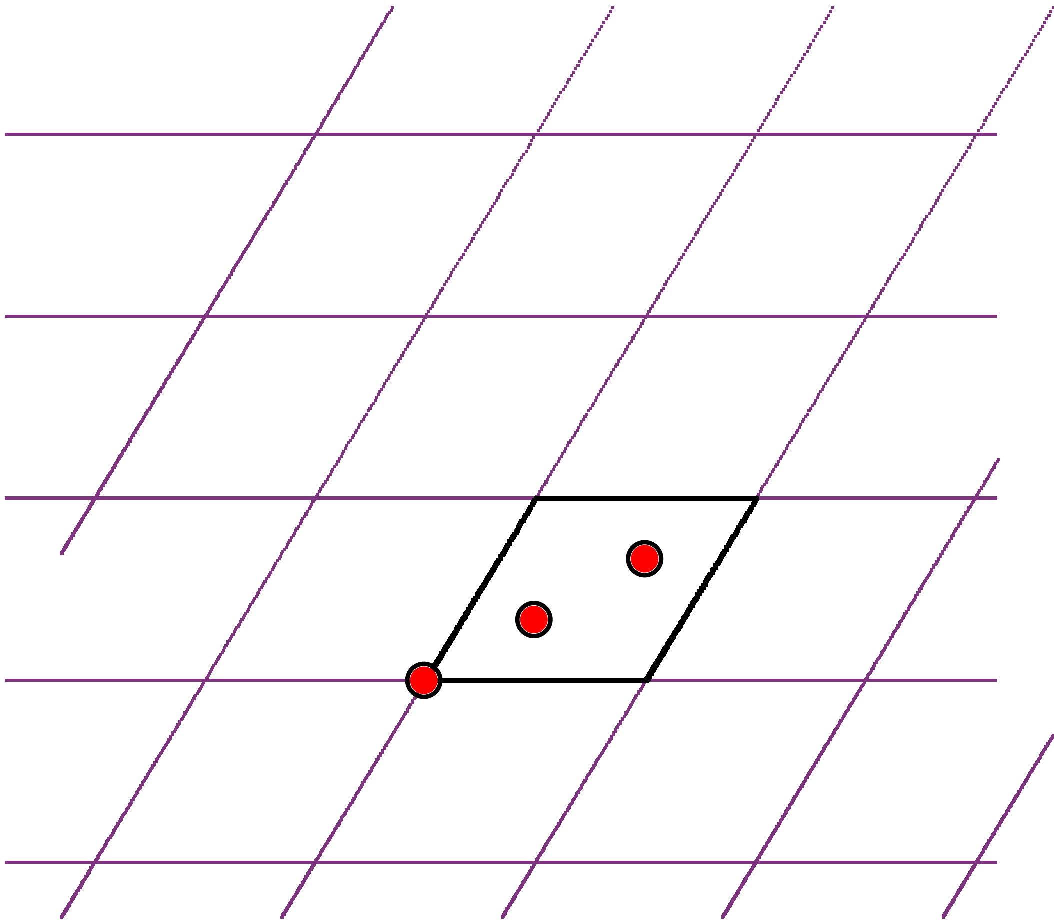
$U(1)$ CS on $T^2$ at level $k$

has $k$ independent states

$\upsilon = \pi/3 \Rightarrow 1$ state.

## Number of states for $U(1)$ $\tau = i\pi/3$ $(k = 3)$

(rotation by $\upsilon = 2\pi/3$)



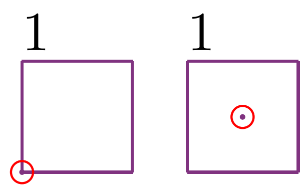
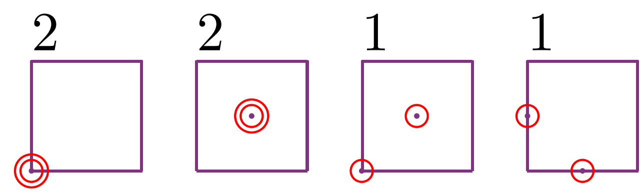
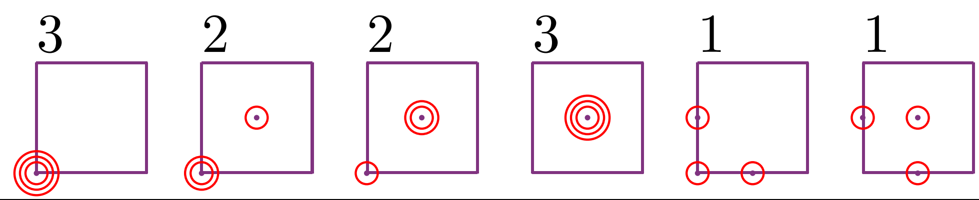
$U(1)$ CS on $T^2$ at level $k$

has $k$ independent states

$\upsilon = 2\pi/3 \Rightarrow 3$ state.

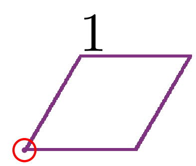
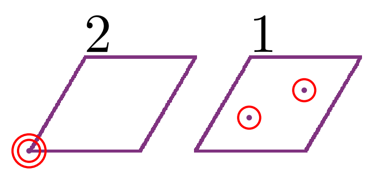
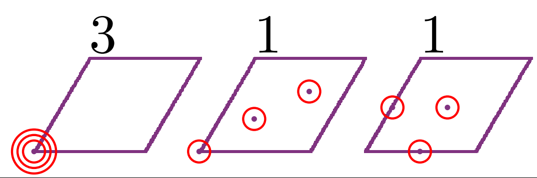
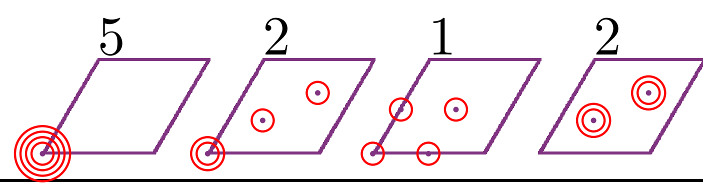
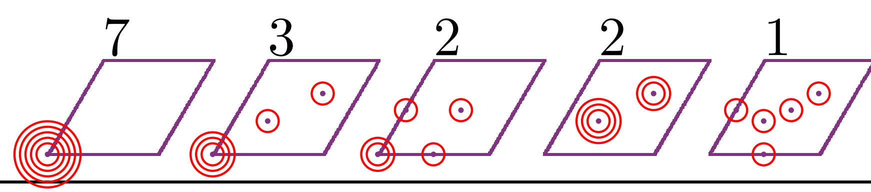
# Counting number of ground states

(Singlet RR ground state)

| | | | |
|---|---|---|---|
| $v = \frac{\pi}{2}$ | $n = 1$ |  | $N_s = 2$ |
| | $n = 2$ |  | $N_s = 6$ |
| | $n = 3$ |  | $N_s = 12$ |

# Counting number of ground states

(Singlet RR ground state)

| | | | |
|---|---|---|---|
| $v = \frac{\pi}{3}$ | $n = 1$ |   | $N_s = 1$ |
| | $n = 2$ |   | $N_s = 3$ |
| | $n = 3$ |   | $N_s = 5$ |
| | $n = 4$ |   | $N_s = 10$ |
| | $n = 5$ |   | $N_s = 15$ |

| Number of states for $U(n)$ on $T^2$ | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | $n$ | | | | |
| $\tau$ | $\upsilon$ | $\vert k\vert$ | 1 | 2 | 3 | 4 | 5 |
| $e^{i\pi/3}$ | $\frac{\pi}{3}$ | 1 | **1** | **3** | **5** | **10** | **15** |
| $i$ | $\frac{\pi}{2}$ | 2 | **2** | **6** | **12** | | |
| $e^{i\pi/3}$ | $-\frac{2\pi}{3}$ | 3 | **3** | **9** | | | |

| Number of states for $U(n)$ on $T^2$ | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | $n$ | | | | |
| $\tau$ | $\upsilon$ | $\|k\|$ | 1 | 2 | 3 | 4 | 5 |
| $e^{i\pi/3}$ | $\frac{\pi}{3}$ | 1 | **1** | **3** | **5** | **10** | **15** |
| $i$ | $\frac{\pi}{2}$ | 2 | **2** | **6** | **12** | | |
| $e^{i\pi/3}$ | $-\frac{2\pi}{3}$ | 3 | **3** | **9** | | | |

Chern-Simons:

$U(1)$ level $k$: $N_s = k$.

$SU(2)$ level $k$: $N_s = k + 1$.

$SU(3)$ level $k$: $N_s = (k+1)(k+2)/2$.

$U(n)_k \to [U(1)_{kn} \times SU(n)_k]/\mathbb{Z}_n \to$ multiply $N_s$ of $SU(n)$ by $k/n$.

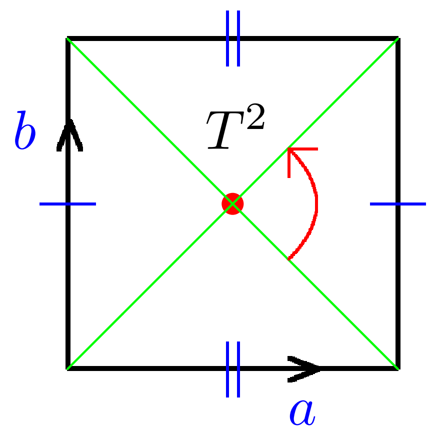
# $\mathbb{Z}/2\mathbb{Z}$ winding number

$$\begin{array}{ccc} T^2(\tau) & \longrightarrow & X \\ & & \downarrow \\ & & S^1 \end{array}$$

$$H_1(X) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

$$H_1(S^1) \qquad \{0, a\}$$

$$[x]$$

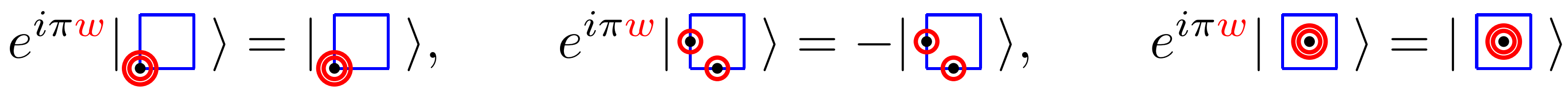



$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ -a \end{pmatrix}$$

Homology of string configuration: $nx + wa, \quad w \in \mathbb{Z}/2\mathbb{Z}$

For $n = 1$:

For $n = 2$:

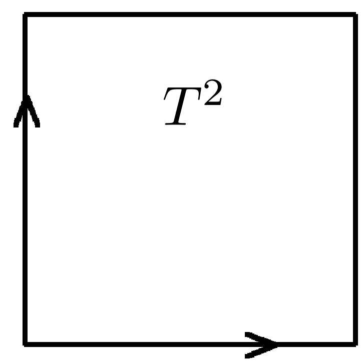


# $\mathbb{Z}/2\mathbb{Z}$ momentum

$$\begin{tikzcd} T^2(\tau) \arrow[r] & X \arrow[d] \\ & S^1 \end{tikzcd}$$

$e^{i\pi p}$ = translation in the fiber.

$p \in \mathbb{Z}/2\mathbb{Z}$



# $\mathbb{Z}/2\mathbb{Z}$ momentum

$$\begin{array}{ccc} T^2(\tau) & \longrightarrow & X \\ & & \downarrow \\ & & S^1 \end{array}$$

$e^{i\pi p}$ = translation in the fiber.

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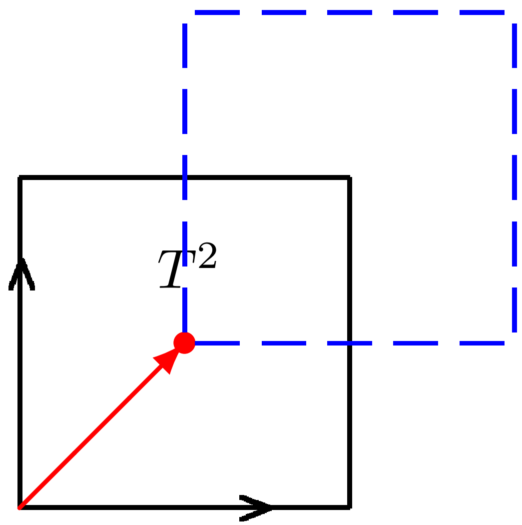
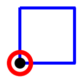
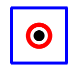
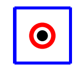
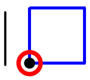
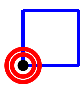
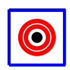
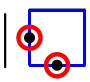
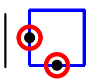
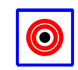
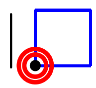


For $n = 1$:

$e^{i\pi p}|$  $\rangle = |$  $\rangle$, $\qquad e^{i\pi p}|$  $\rangle = -|$  $\rangle$.

For $n = 2$:

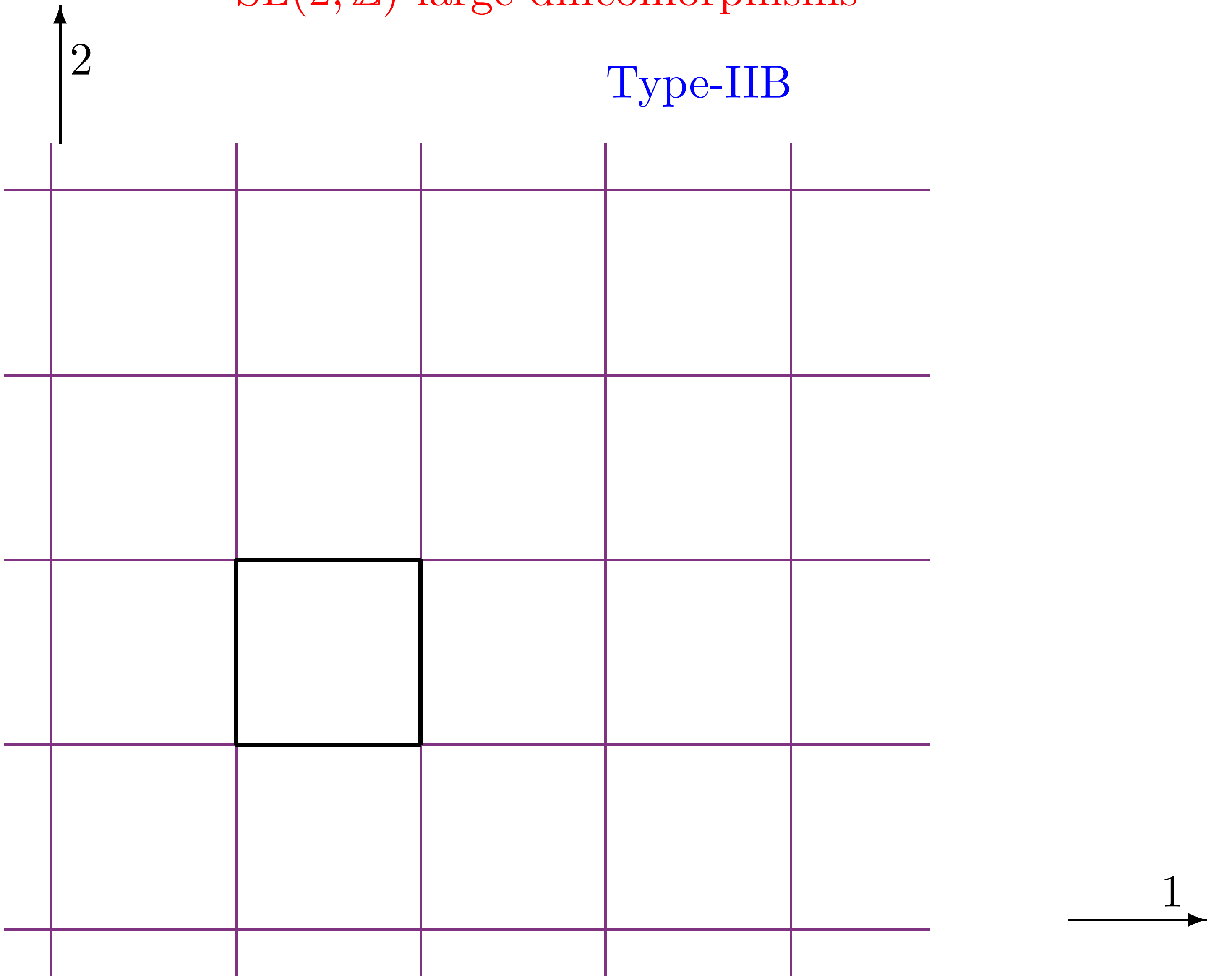
$e^{i\pi p}|$  $\rangle = |$  $\rangle$, $\qquad e^{i\pi p}|$  $\rangle = -|$  $\rangle$, $\qquad e^{i\pi p}|$  $\rangle = |$  $\rangle$.

$$e^{i\pi p} e^{i\pi w} = (-1)^n e^{i\pi w} e^{i\pi p}$$

In $n = 1$ Chern-Simons theory:

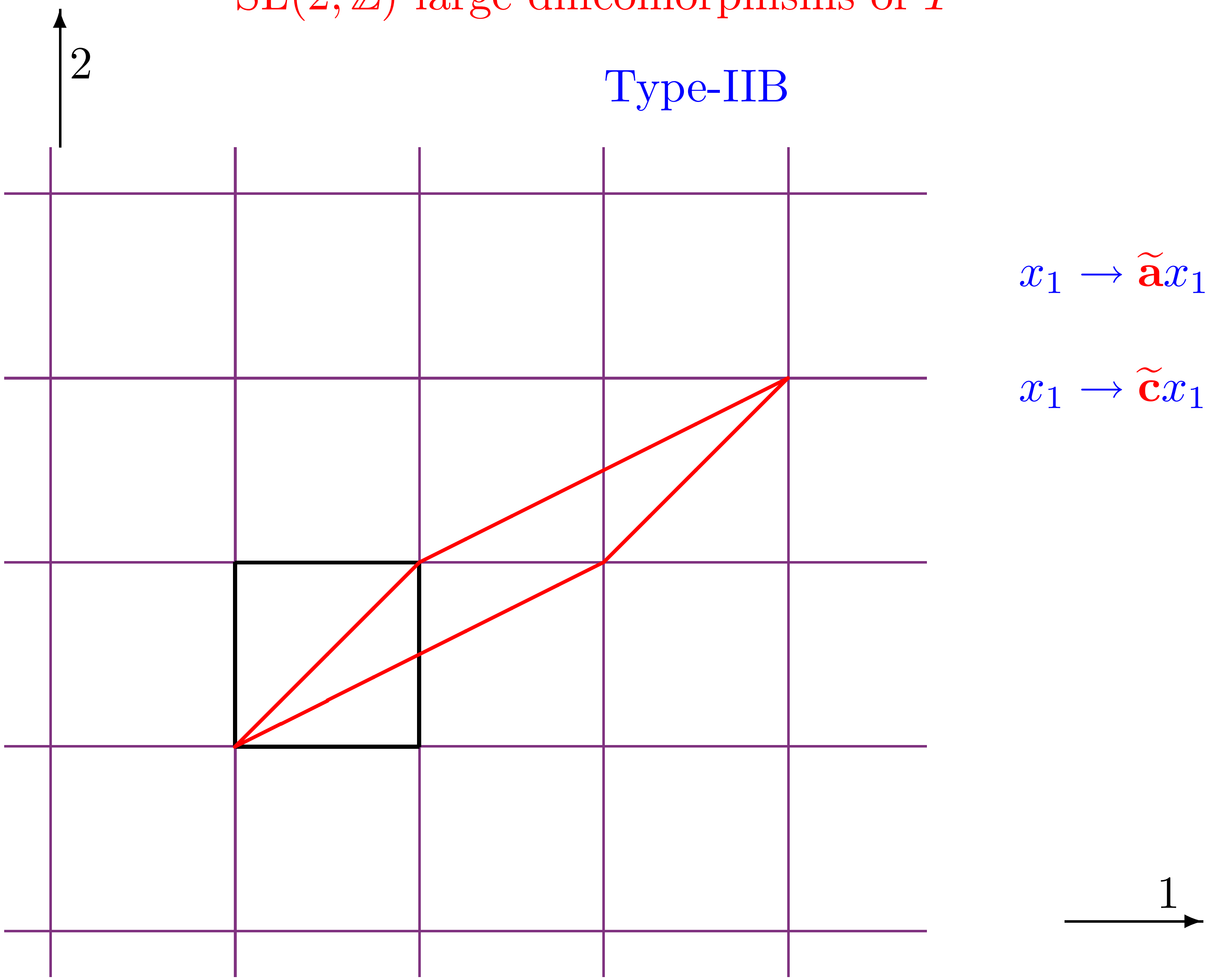
$$e^{i\pi p} \longrightarrow e^{\oint_{a'} A}, \qquad e^{i\pi w} \longrightarrow e^{\oint_{b'} A}.$$

SL$(2,\mathbb{Z})$ large diffeomorphisms



## $\mathrm{SL}(2,\mathbb{Z})$ large diffeomorphisms of $T^2$

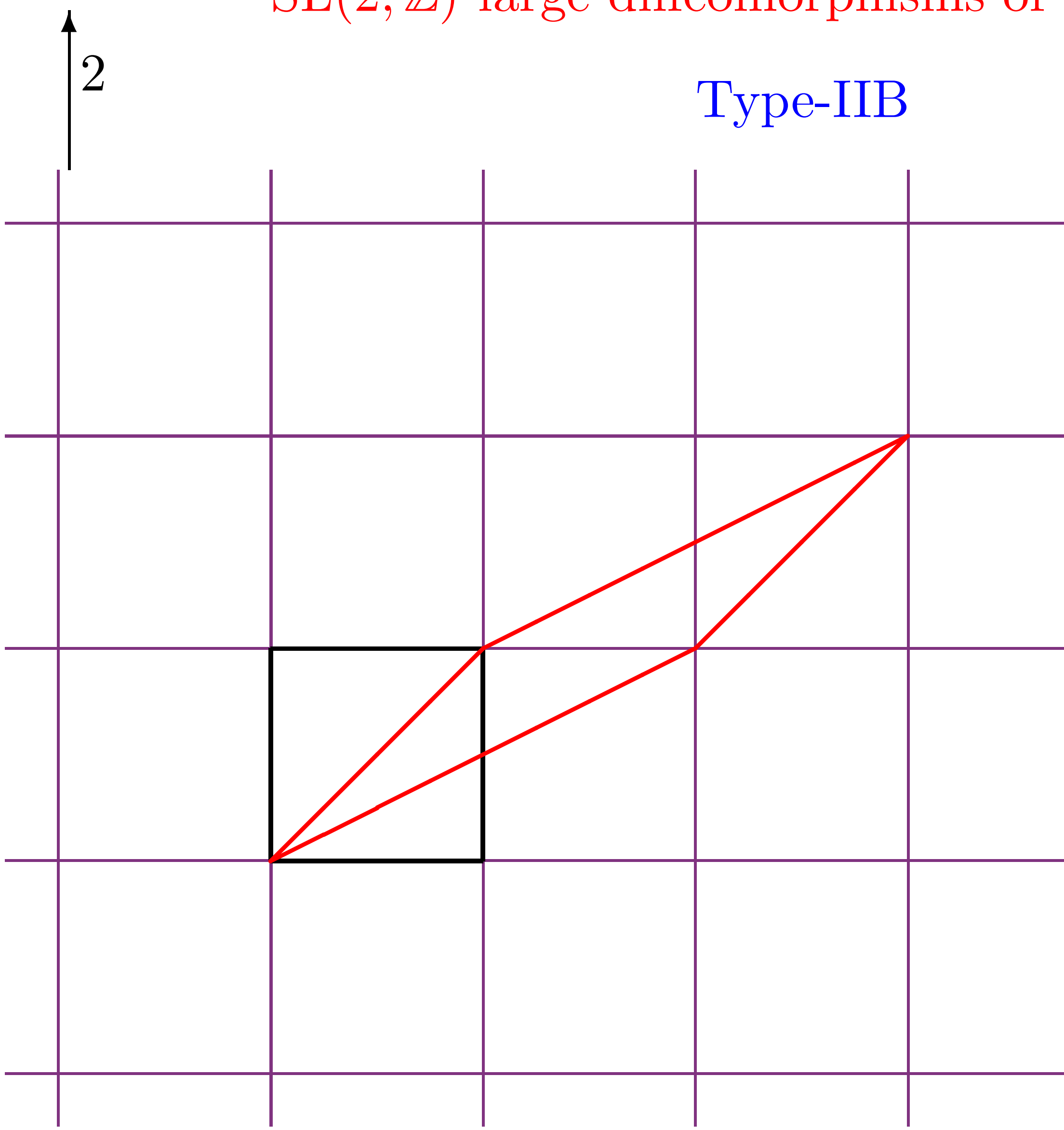
### Type-IIB




$$x_1 \to \widetilde{\mathbf{a}} x_1 + \widetilde{\mathbf{b}} x_2$$

$$x_1 \to \widetilde{\mathbf{c}} x_1 + \widetilde{\mathbf{d}} x_2$$

## $\mathrm{SL}(2,\mathbb{Z})$ large diffeomorphisms of $T^2$

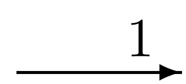



$$x_1 \to \widetilde{\mathbf{a}} x_1 + \widetilde{\mathbf{b}} x_2$$

$$x_1 \to \widetilde{\mathbf{c}} x_1 + \widetilde{\mathbf{d}} x_2$$

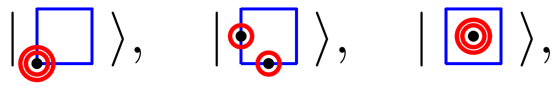
Type-IIA

$\Rightarrow$T-duality!

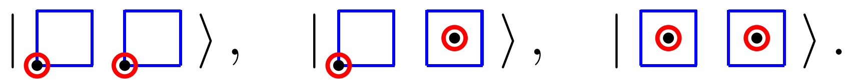


# Example: results for $n = 2$ $(\tau \to -1/\tau)$

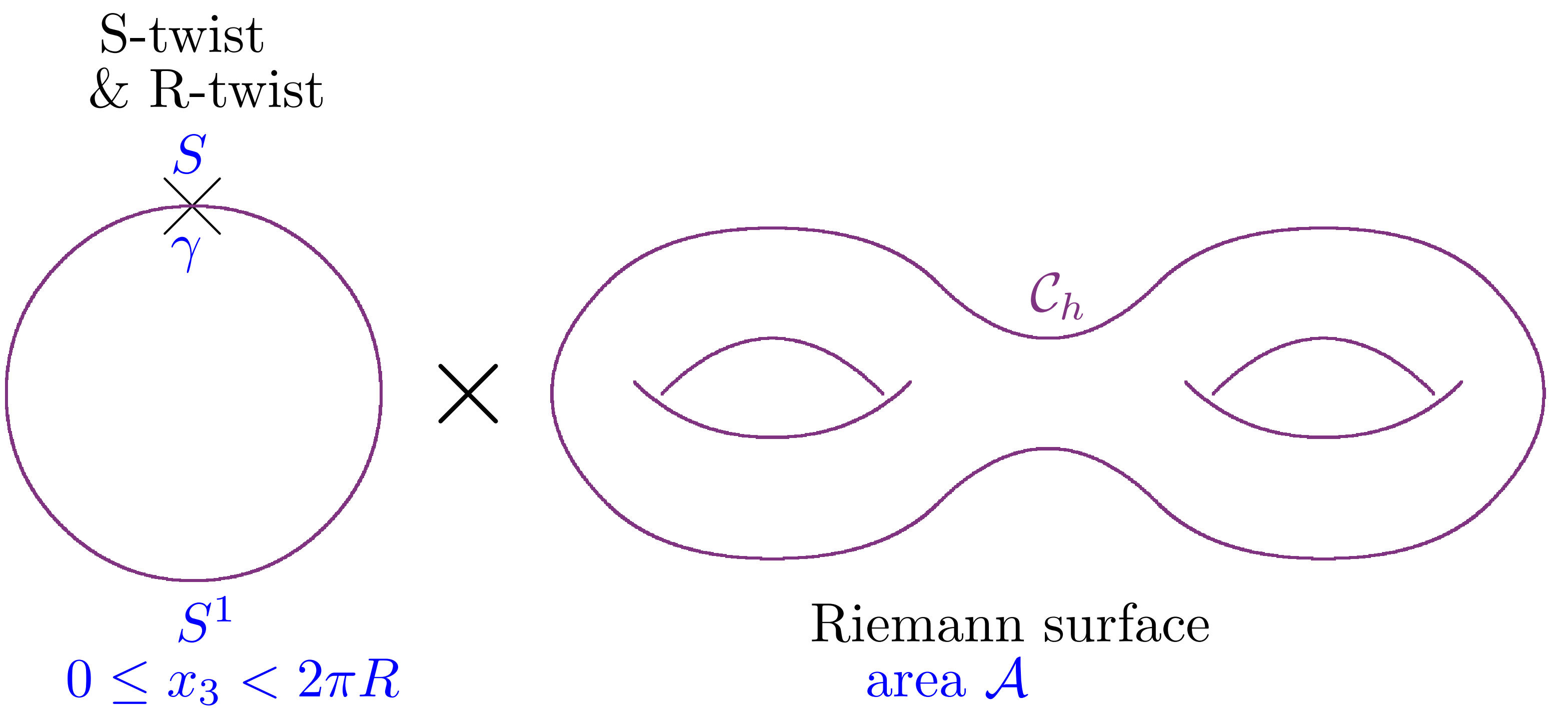
3 single-particle states:



3 two-particle states:



Two copies of $U(2)_2$ CS.




$$\mathcal{A} \to 0 \Longrightarrow \sigma\text{-model on } \mathcal{M}_H$$

S-duality becomes T-duality [Harvey & Moore & Strominger; Bershadsky & Johansen & Sadov & Vafa]

# Witten Index

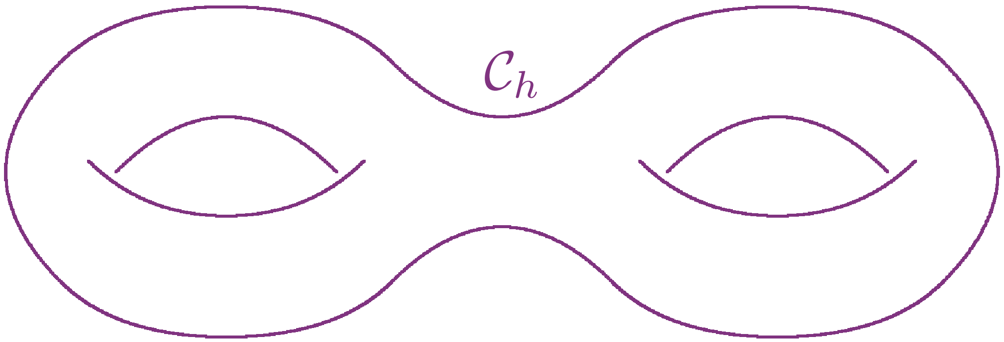
$$\#\{\text{vacua of 2+1D theory on } \mathcal{C}_h\} = I = \mathrm{tr}_0\{(-1)^F \mathcal{T}(\mathbf{s})\gamma\}\,.$$

# Hitchin's equations

$$\boxed{\begin{aligned} &F_{z\overline{z}} = [\phi_z, \overline{\phi}_{\overline{z}}] \\ &D_z\overline{\phi}_{\overline{z}} = D_{\overline{z}}\phi_z = 0 \end{aligned}}$$

$A_z$ gauge field

$\phi_z$ adj.-valued 1-form




Riemann surface

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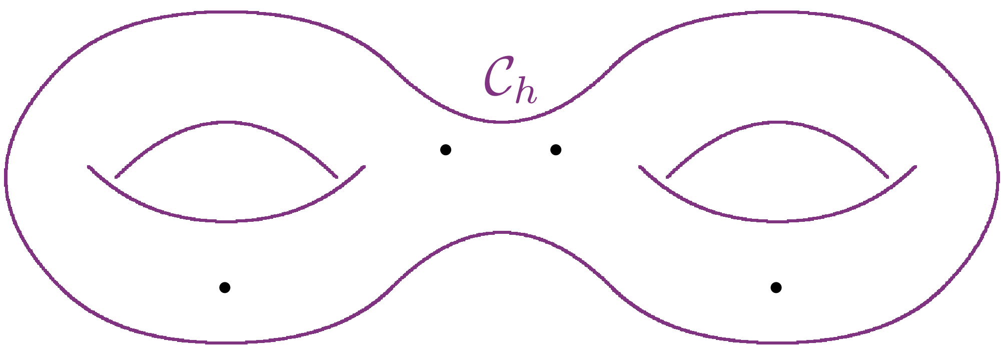
$b_{zz} = \mathrm{tr}(\phi_z^2)$ holomorphic with $4h-4$ zeroes.

Space of quadratic differentials: $\mathbb{C}^{3(h-1)}$

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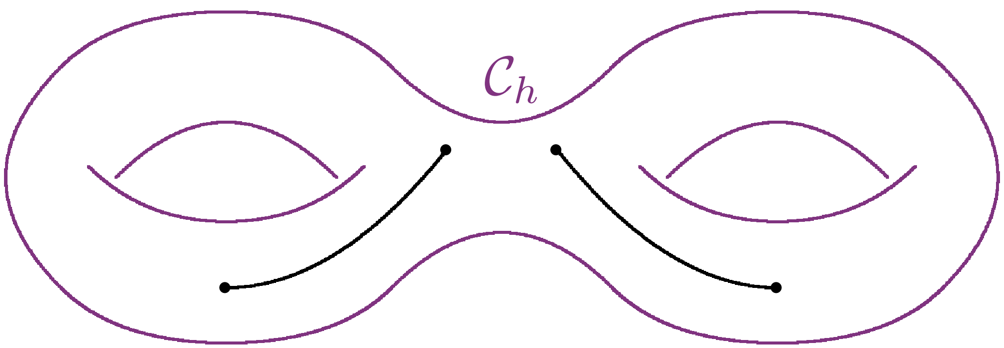
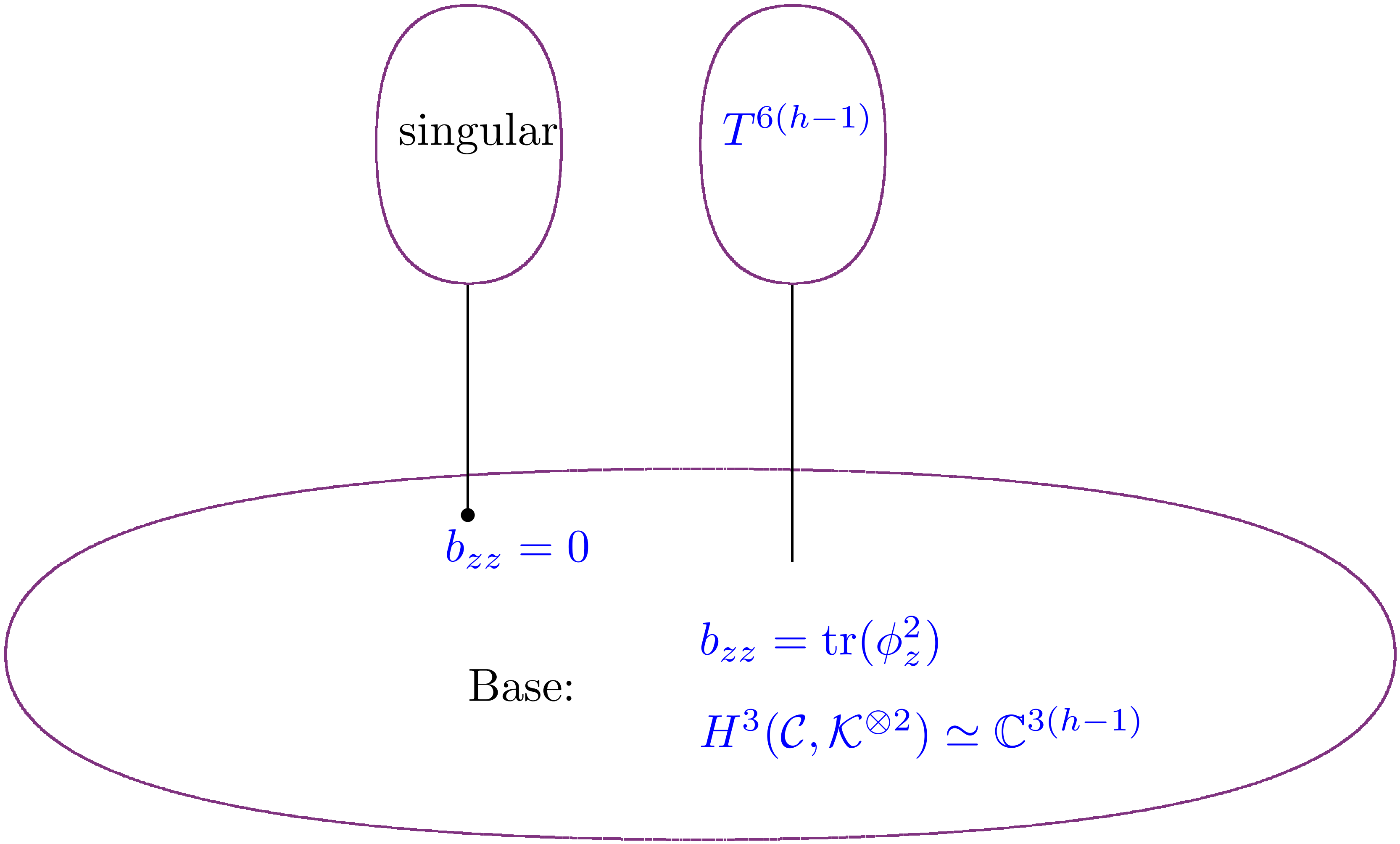



Riemann surface

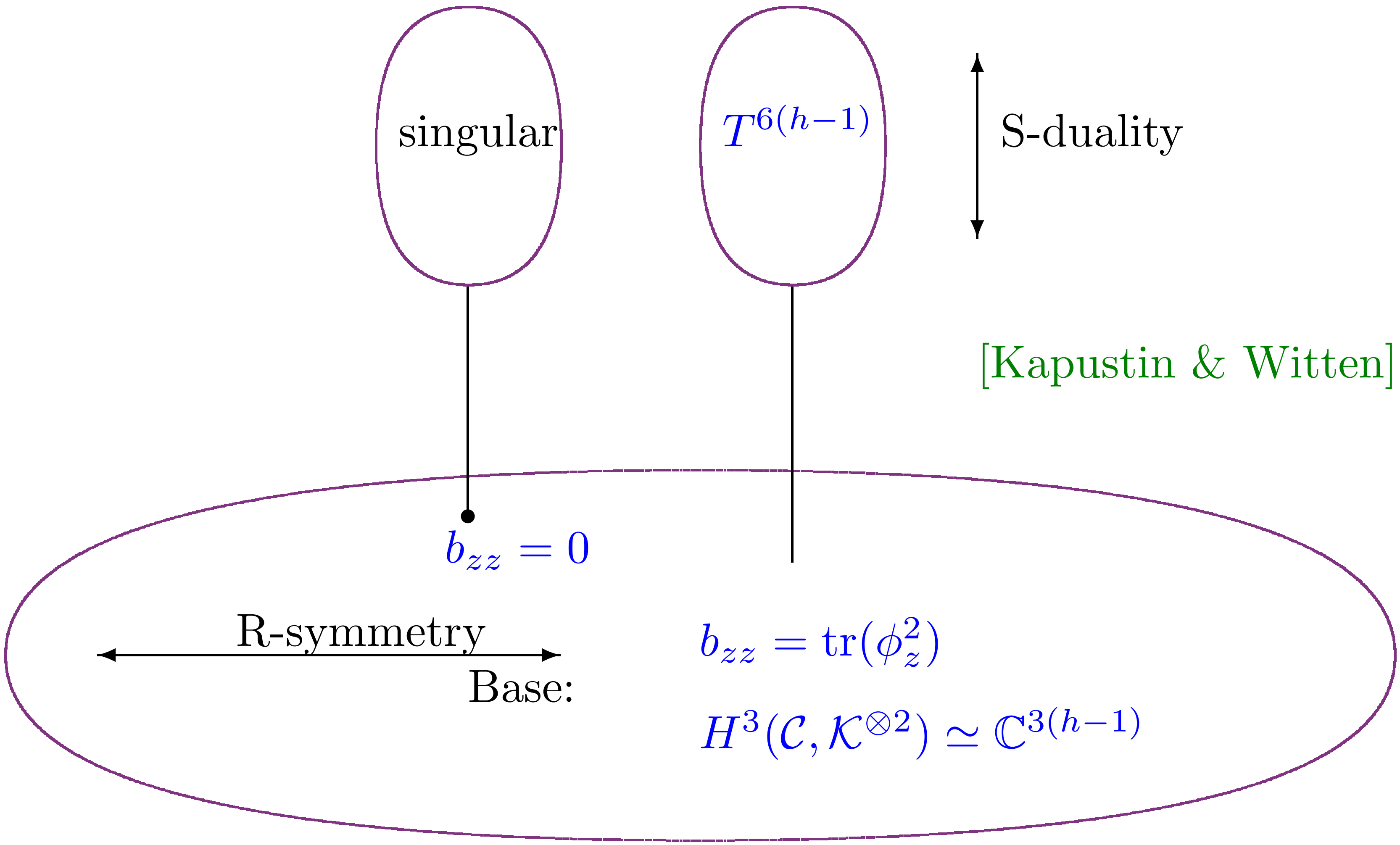
Double cover has genus $4h - 3$

Prym subspace of its Jacobian: $T^{6(h-1)}$

# Hitchin Fibration



# Hitchin Fibration



# The fiber over $b_{zz} = 0$ . . .

$$b_{zz} = \mathrm{tr}(\phi_z^2) = 0$$

Case 1: $\phi_z = 0 \Longrightarrow \mathcal{M}_{\mathrm{fc}}$ = moduli space of flat connections.

Case 2: $\phi_z = \begin{pmatrix} 0 & \alpha_z \\ 0 & 0 \end{pmatrix}$ , $\qquad A_{\overline{z}} = \begin{pmatrix} a_{\overline{z}} & c_{\overline{z}} \\ 0 & -a_{\overline{z}} \end{pmatrix}$ ,

$$a_{\overline{z}} = -\frac{1}{2}\partial_{\overline{z}} \log \alpha_z \; ,$$

$$\partial_z a_{\overline{z}} - \partial_{\overline{z}} a_z = |\alpha_z|^2 + |c_{\overline{z}}|^2, \qquad \text{and } \frac{c_{\overline{z}}^*}{\alpha_z} = \text{holomorphic.}$$

Special subcase of 2: $c_{\overline{z}} = 0$.

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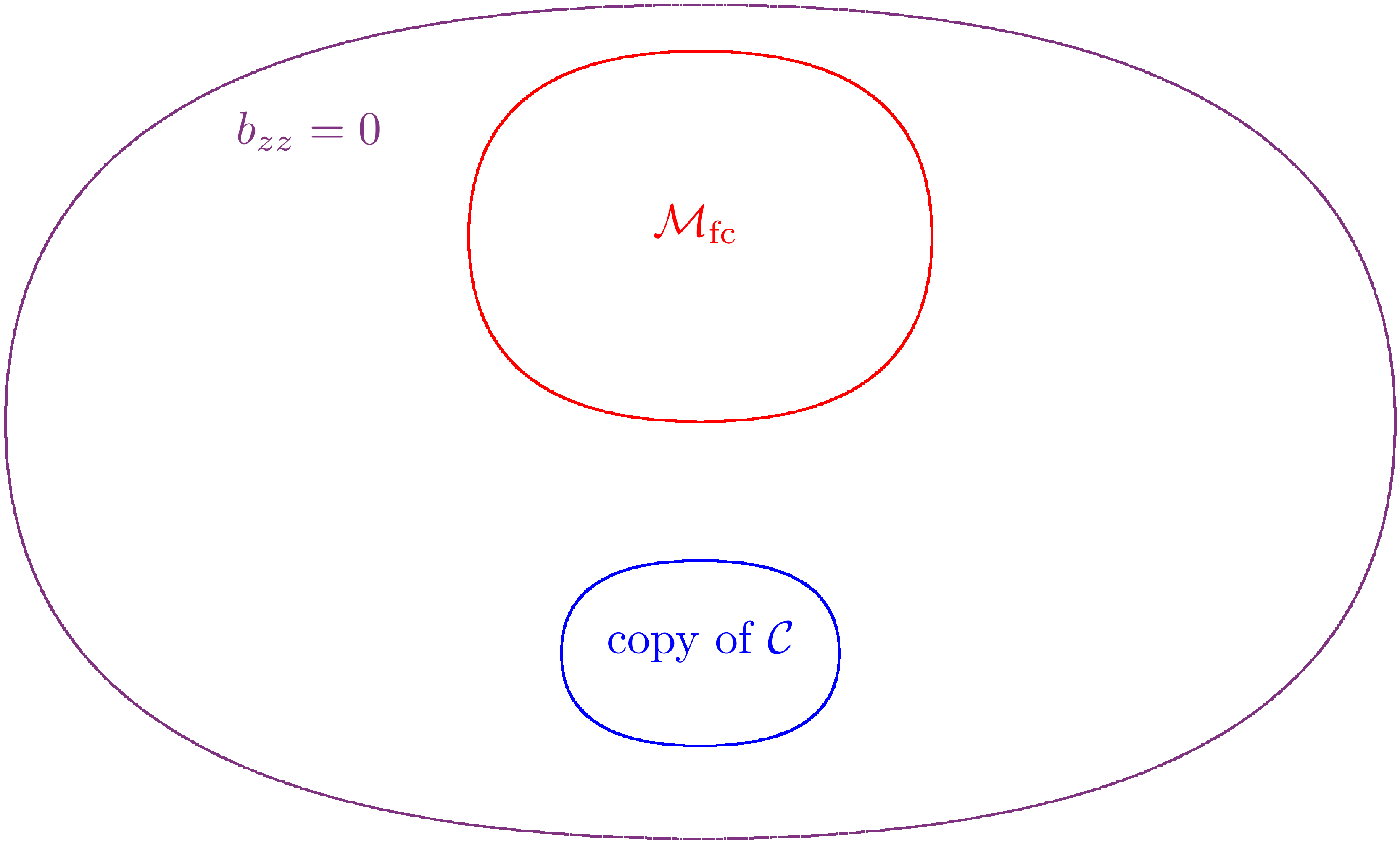
Special subcase of 2: $c_{\overline{z}} = 0$.

if also genus $h = 2$: $\alpha_z$ has a single simple zero on $\mathcal{C}_2$ which determines the solution uniquely up to gauge.

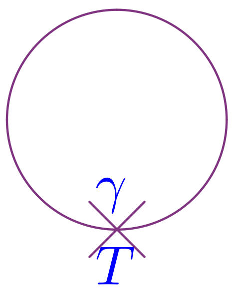
# The fiber over $b_{zz} = 0$ . . .



# T-duality and Geometric Quantization

1+1D $\sigma$-model with target space $X$

$T$ = T-duality (mirror symmetry) twist
$\gamma$ = some isometry twist

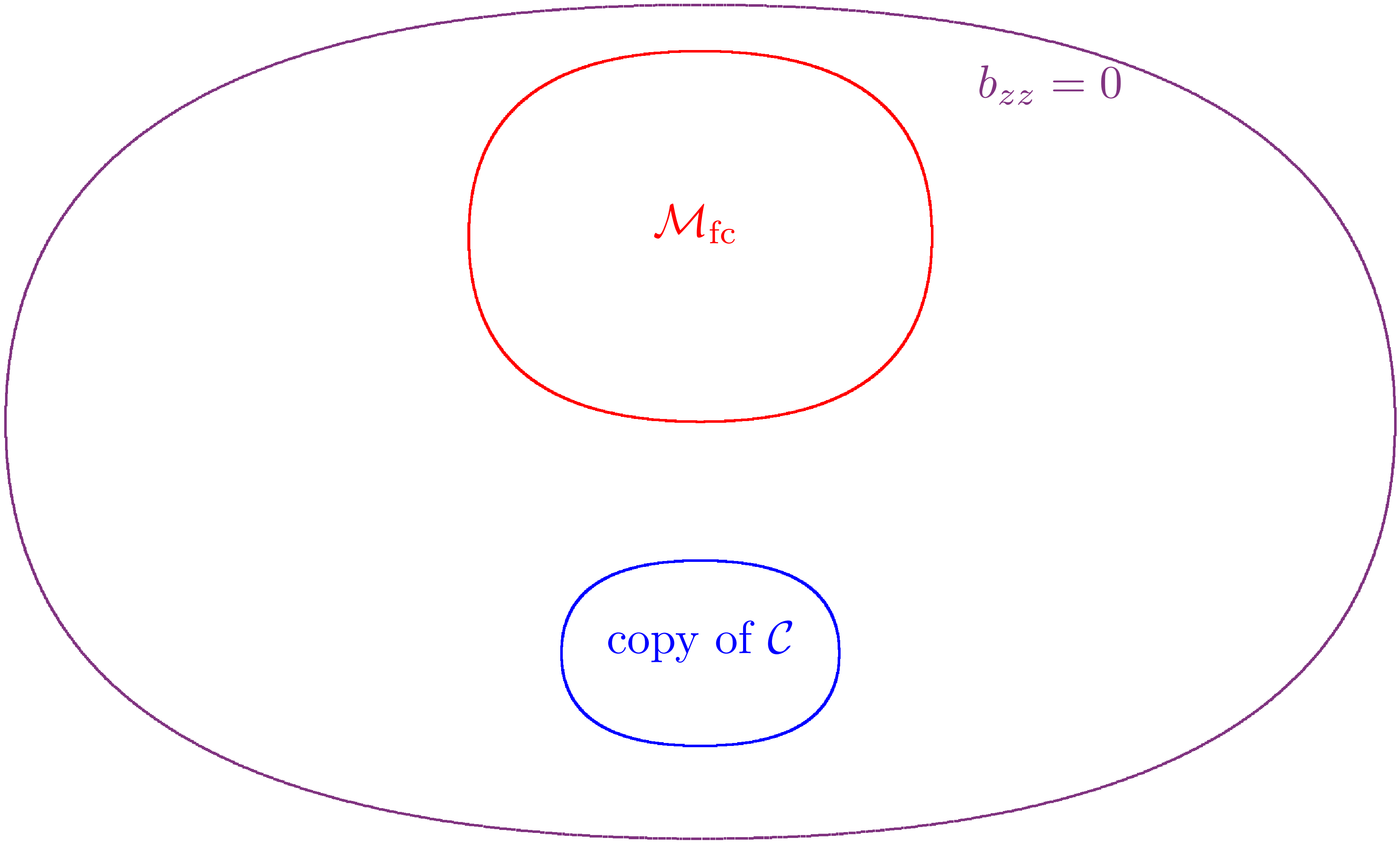



IR?

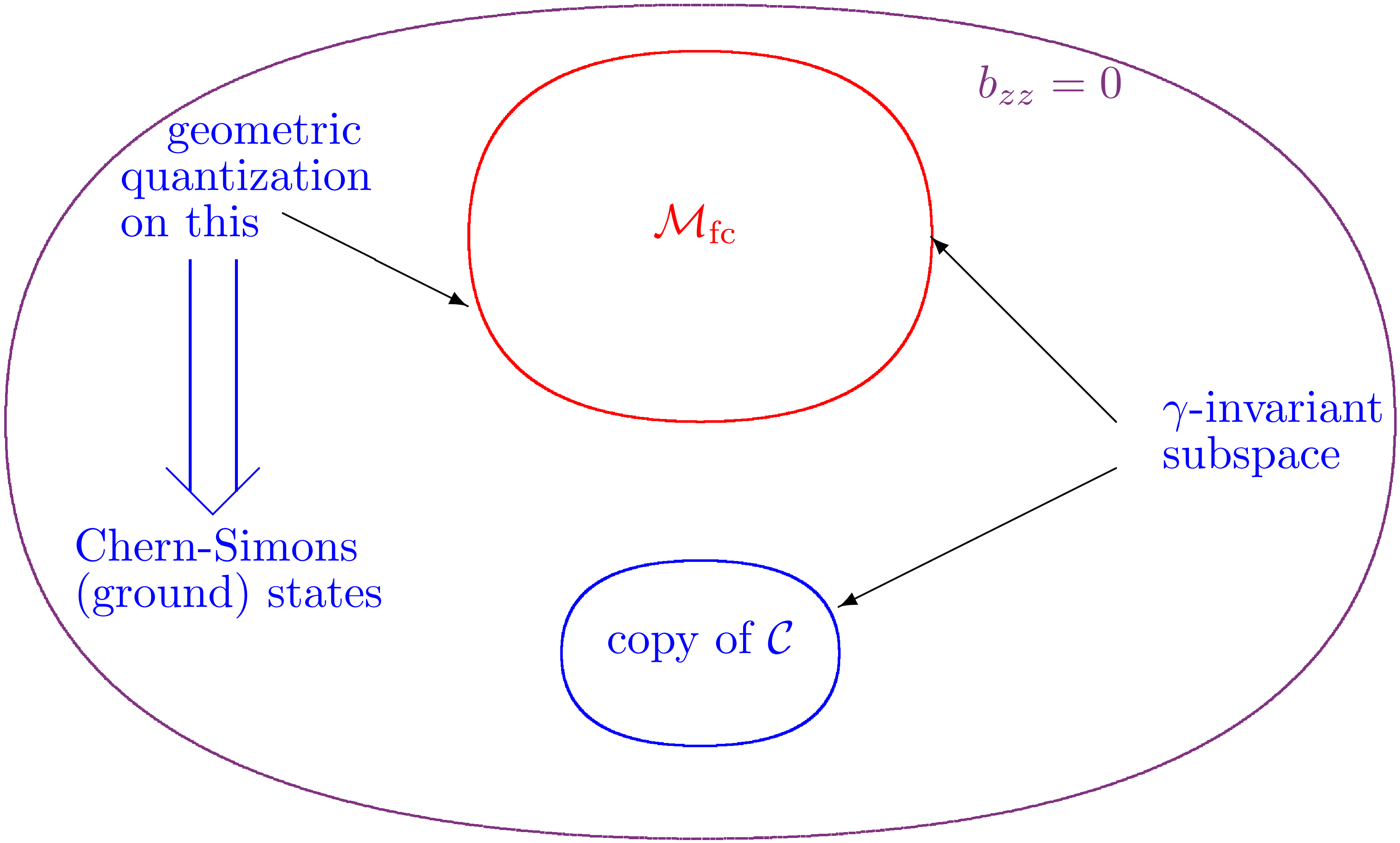
Geometric quantization on $\gamma$-invariant subspace???

[cf. Gukov & Witten]

# **The fiber over $b_{zz} = 0$ ......**



# The fiber over $b_{zz} = 0$ ......



# Conclusions

- Compactification of $N = 4\ U(n)$ SYM on $S^1$ with an S-duality twist, at a self-dual $\tau$ seems to give a topological 2+1D QFT in IR for $n$ sufficiently small;
- Number of (ground) states on $T^2$ can be computed by string dualities;
- Number of (ground) states on $\mathcal{C}_h$ ($h > 1$) could be computed if we could determine the signs in the action of S-duality on $H^*(\mathcal{M}_H)$;

# Open questions

- What is this topological 2+1D theory?
- Wilson lines?
- $n \geq 4$ and ABJM theory?
- Mirror symmetry twist and geometric quantization?
- Nonlocal topological structure from the kernel $\mathcal{S}(A, A_D)$?
- Can we extract any new clues about S-duality from this?

## Thank you!

## Title