Mass Terms in Twistor String Theory

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BASED ON:

- “Massless and massive three dimensional super Yang-Mills theory and mini-twistor string theory,” [arXiv:hep-th/0502076]
by D. W. Chiou, OG, Y. P. Hong, B. S. Kim and I. Mitra.

- “A Deformation of Twistor Space and a Chiral Mass Term,” [arXiv:hep-th/0510???]
by D. W. Chiou, OG and B. S. Kim.

Introduction

1. Brief review of twistor and minitwistor spaces.

2. Witten's observations on $N = 4$ supersymmetric Yang-Mills theory.

3. Mass terms to the scalars and fermions.

4. Dimensional reduction to $D = 3$.

5. Physical interpretation in $D = 3$ of Witten's holomorphic curves.

6. More mass terms in $D = 3$.

Twistor space [review]

I will begin with a review of twistor space, mostly following [Paul Baird].

The twistor transform uncovers holomorphic structure underlying free field equations.

Simplest example:
A harmonic function on $\mathbb{R}^2$:

$$\begin{aligned} 0 &= \triangle\phi := \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} \\ \Longrightarrow & \quad \phi(x_1, x_2) = f(x_1 + ix_2) + g(x_1 - ix_2). \end{aligned}$$

Set

$$z := x_1 + ix_2, \qquad \overline{z} := x_1 - ix_2.$$

Twistor space for $\mathbb{R}^2$ is $\mathbb{C} \cup \mathbb{C}$, the complex z-plane together with the complex $\overline{z}$-plane.

Harmonic functions on $\mathbb{R}^3$ [review]

There is a map from harmonic functions on $\mathbb{R}^3$ to holomorphic functions on minitwistor space $T\mathbb{CP}^1$. [Hitchin 1982]

Again, I will follow [Baird]. We are looking for solutions to

$$0 = \triangle\phi := \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2}.$$

For fixed θ, and any holomorphic function f on $\mathbb{C}$,

$$0 = \triangle f(x_1 + ix_2 \sin\theta + ix_3 \cos\theta).$$

We can construct more complicated harmonic functions by [Whittaker]

$$\phi(x_1, x_2, x_3) = \int_0^{2\pi} F(x_1 + ix_2 \sin\theta + ix_3 \cos\theta,\ \theta) d\theta$$

Spherical coordinates [review]

Let's change to a basis of spherical harmonics. Define spherical coordinates

$$x_1 = r\cos u, \quad x_2 = r\sin u \sin v, \quad x_3 = r\sin u \cos v.$$

Then, for $l \geq 0$, and $|m| \leq l$, the spherical harmonics can be written as

$$r^l Y_{lm}(u,v) = \frac{\sqrt{(2l+1)(l-m)!(l+m)!}}{4\pi^{3/2} i^{3m} l!} \times \int_0^{2\pi} d\theta\, \underbrace{e^{im\theta}(x_1 + ix_2 \sin\theta + ix_3 \cos\theta)^l}_{F},$$

Minitwistor space [review]

Whittaker's formula had the following integrand:

$$F(x_1 + ix_2 \sin\theta + ix_3 \cos\theta,\, \theta)$$

It is more convenient to change variables

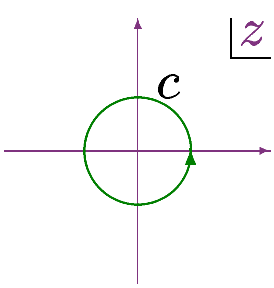
$$w = 2e^{i\theta}(x_1 + ix_2 \sin\theta + ix_3 \cos\theta), \qquad z = e^{i\theta}.$$

Given the analytic function F, it is convenient to define a related analytic function φ by

$$\varphi(e^{i\theta}, w) := e^{-i\theta} F(\tfrac{1}{2}e^{-i\theta} w, \theta).$$

We assume that we can extend φ to an analytic function $\varphi(z, w)$ defined in a neighborhood of the circle $|z| = 1$. Whittaker's formula can now be rewritten as

$$\boxed{\phi(\vec{x}) = \frac{1}{2\pi i} \oint_c \varphi(z, -[x_2 - ix_3] + 2zx_1 + z^2[x_2 + ix_3])dz.}$$



Mintwistor space and $T\mathbb{CP}^1$ [review]

$$\phi(\vec{x}) = \frac{1}{2\pi i}\oint_c \varphi(z, \underbrace{-[x_2 - ix_3] + 2zx_1 + z^2[x_2 + ix_3]}_{w})dz.$$

The contour integral for ϕ does not change if we modify

$$\begin{aligned} \varphi(z,w) \quad \longrightarrow \quad & \varphi(z,w) + [\text{holomorphic at } z = 0] \\ & +[\text{holomorphic at } z = \infty] \end{aligned}$$

provided we define the good holomorphic coordinates at $z = \infty$ as

$$z' = \frac{1}{z}, \qquad w' = \frac{w}{z^2}.$$

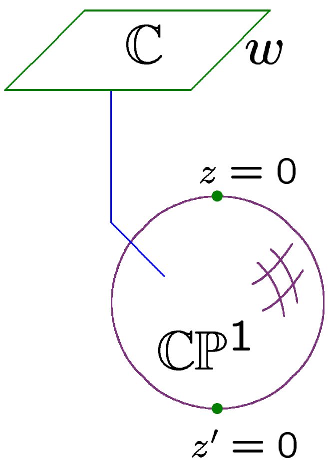
Then, a holomorphic function

$$\varphi(z,w) := z'^2 \varphi'(z', w')$$

will give

$$\phi(\vec{x}) = \frac{1}{2\pi i}\oint_c \varphi'(\tfrac{1}{z}, -\tfrac{1}{z^2}[x_2 - ix_3] + \tfrac{2}{z}x_1 + [x_2 + ix_3]\})\frac{dz}{z^2} = 0.$$

Picture of Minitwistor space



$T\mathbb{CP}^1$

Minitwistor space is the tangent space of $\mathbb{CP}^1$.

Two patches:

$z' \neq 0$	$z \neq 0$
z	$z' = 1/z$
w	$w' = w/z^2$

Minitwistor transform

So, the minitwistor transform $\varphi(z, w)$ transforms like a meromorphic section of the line-bundle $\mathcal{O}(-2)$ over $T\mathbb{CP}^1$.

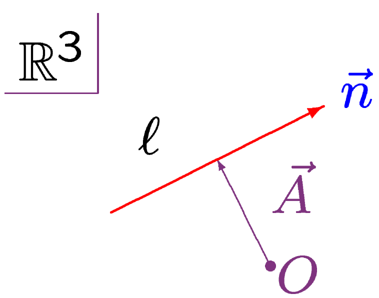
It is defined up to local holomorphic sections at $z = 0$ and $z = \infty$.

$\Longrightarrow$ It is an element of the sheaf cohomology $H^1(T\mathbb{CP}^1, \mathcal{O}(-2))$.

Geometrical interpretation [review]

Minitwistor space has a simple geometrical interpretation {that I learned from P. Baird's review}:

$T\mathbb{CP}^1$ is the space of oriented lines in $\mathbb{R}^3$.



$$z = \frac{n^2 - in^3}{1 + n^1} \in \mathbb{CP}^1 \simeq \mathbb{C} \cup \{\infty\}$$

$$w = \frac{-(1 + n^1)(A^2 - iA^3) + (n^2 - in^3)A^1}{(1 + n^1)^2}$$

(By stereographic projection.)

Twistor space [review]

All this also works for $\mathbb{R}^4$. We are looking for harmonic functions

$$0 = \triangle\phi := \sum_{k=1}^{4} \frac{\partial^2 \phi}{\partial x_k^2}$$

Any holomorphic function of two complex variables of the form

$$f(x_1 + ix_2,\, x_3 + ix_4)$$

is harmonic. This is also true for any other choice of complex structure. A generic choice of complex structure is described by $[\lambda^1, \lambda^2] \in \mathbb{CP}^1$. For this complex structure we require that

$$\lambda^1 \begin{pmatrix} x_1 + ix_2 \\ x_3 + ix_4 \end{pmatrix} + \lambda^2 \begin{pmatrix} -x_3 + ix_4 \\ x_1 - ix_2 \end{pmatrix}$$

be analytic. Then

$$\begin{aligned} f_{[\lambda^1,\lambda^2]}(x_1,x_2,x_3,x_4) := f(&\lambda^1[x_1 + ix_2] + \lambda^2[-x_3 + ix_4], \\ &\lambda^1[x_3 + ix_4] + \lambda^2[x_1 - ix_2]). \end{aligned}$$

is harmonic.

Spinor notation

We can get any harmonic function on $\mathbb{R}^4$ by integrating over different complex structures

$$\phi(x) = \oint_c f_{[1,z]}(x; z) dz.$$

We set

$$(x_{\alpha\dot{\alpha}}) \equiv \begin{pmatrix} x_{1\dot{1}} & x_{1\dot{2}} \\ x_{2\dot{1}} & x_{2\dot{2}} \end{pmatrix} := \begin{pmatrix} x_1 + ix_2 & -x_3 + ix_4 \\ x_3 + ix_4 & x_1 - ix_2 \end{pmatrix}.$$

$$\alpha = 1, 2, \quad \dot{\alpha} = \dot{1}, \dot{2}, \quad x_{\alpha\dot{\alpha}} = x^\mu \sigma_{\mu\alpha\dot{\alpha}}.$$

Then,

$$\phi(x) = \oint_c dz f(\mu_{\dot{\alpha}}, \lambda) \Big|_{\mu_{\dot{\alpha}} = -x_{\alpha\dot{\alpha}}\lambda^\alpha}$$

$$\lambda^1 \equiv 1, \lambda^2 \equiv z.$$

Twistors and shockwaves

For $\mathbb{R}^{2,2}$ the twistors are real.

$$(x_{\alpha\dot\alpha}) \equiv \begin{pmatrix} x_{1\dot 1} & x_{1\dot 2} \\ x_{2\dot 1} & x_{2\dot 2} \end{pmatrix} := \begin{pmatrix} x_1 + x_2 & -x_3 + x_4 \\ x_3 + x_4 & x_1 - x_2 \end{pmatrix},$$

and the shockwave

$$\boxed{\Phi_{(\mu,\lambda)}(x^1,\ldots,x^4) \propto \delta^2(x_{\alpha\dot\alpha}\lambda^\alpha + \mu_{\dot\alpha})}$$

solves the wave equation

$$0 = \frac{\partial^2\Phi}{\partial x_1^2} + \frac{\partial^2\Phi}{\partial x_3^2} - \frac{\partial^2\Phi}{\partial x_2^2} - \frac{\partial^2\Phi}{\partial x_4^2}.$$

$\widetilde{t} = (\mu, \lambda)$ denotes a $D = 4$ twistor.

So, instead of working in the usual basis of plane-waves

$$\Phi_k(x) = e^{ik\cdot x}$$

in twistor theory we work in a basis of shockwaves.

Twistor space, for this example, is $\mathbb{RP}^3 \setminus \mathbb{RP}^1$ with projective coordinates

$$Z^1 = \lambda^1, Z^2 = \lambda^2, Z^3 = \mu^{\dot 1}, Z^4 = \mu^{\dot 2},$$
$$(Z^1, Z^2, Z^3, Z^4) \sim (\zeta Z^1, \zeta Z^2, \zeta Z^3, \zeta Z^4),$$
$$(Z^1, Z^2) \neq (0, 0).$$

Background

- Witten discovered remarkable properties of perturbative scattering amplitudes in $N = 4$ SYM in $D = 4$.

- Switching from a basis of plane-waves to a basis of shock-waves (twistors), Witten found that amplitudes vanish unless certain algebraic conditions (on the incoming and outgoing twistors) hold.

- Witten proposed that a topological B-model with target space $\mathbb{CP}^{3|4}$ (super twistor space) reproduces the SYM amplitudes. Certain non-perturbative effects (D1-instantons) are a crucial ingredient.

$N = 4$ Super-Yang-Mills theory

The Lagrangian of $N = 4$ super Yang-Mills theory:

$$g^2\mathcal{L} = \mathrm{tr}\Big\{\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\sum_I D_\mu\Phi^I D^\mu\Phi^I$$
$$-\frac{1}{4}\sum_{I,J}[\Phi^I,\Phi^J]^2 + \sum_A \psi_\alpha^A\sigma^{\mu\,\alpha\dot\beta}\partial_\mu\psi_{A\dot\beta}$$
$$+\sum_{A,B,I}\Big(\Gamma^I_{AB}\Phi^I\psi_\alpha^A\psi^{B\alpha} + \Gamma^{I\,AB}\Phi^I\psi_{A\dot\alpha}\psi_B^{\dot\alpha}\Big)\Big\},$$

symbol	spacetime	$SU(4)_R$
Φ^I	scalars	$\mathbf{6}$
ψ_α^A	(L-)spinors	$\mathbf{4}$
$\psi_{A\dot\alpha}$	(R-)spinors	$\bar{\mathbf{4}}$

Mass terms

The Lagrangian of mass-deformed $N = 4$ super Yang-Mills theory:

$$
\begin{aligned}
g^2\mathcal{L} = \mathrm{tr}\Big\{ & \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\sum_I D_\mu\Phi^I D^\mu\Phi^I \\
& - \frac{1}{4}\sum_{I,J}[\Phi^I,\Phi^J]^2 + \sum_A \psi^A_\alpha \sigma^{\mu\,\alpha\dot{\beta}}\partial_\mu \psi_{A\dot{\beta}} \\
& + \sum_{A,B,I}\left(\Gamma^I_{AB}\Phi^I\psi^A_\alpha\psi^{B\alpha} + \Gamma^{I\,AB}\Phi^I\psi_{A\dot{\alpha}}\psi^{\dot{\alpha}}_B\right) \\
& + \sum_{A,B} M_{AB}\psi^A_\alpha\psi^{B\alpha} + \sum_{A,B} M^{AB}\psi_{A\dot{\alpha}}\psi^{\dot{\alpha}}_B \\
& + (m^2)_{IJ}\Phi^I\Phi^J \Big\},
\end{aligned}
$$

symbol	spacetime	$SU(4)_R$
Φ^I	scalars	$\mathbf{6}$
ψ^A_α	(L-)spinors	$\mathbf{4}$
$\psi_{A\dot{\alpha}}$	(R-)spinors	$\overline{\mathbf{4}}$
M^{AB}	-	$\mathbf{10}$
M_{AB}	-	$\overline{\mathbf{10}}$

Notation

- Shock-waves on $\mathbb{R}^{2,2}$ (or $\mathbb{C}^4$):

$$\boxed{\Phi_{(\mu,\lambda)}(x^1,\ldots,x^4) \propto \delta^2(x_{\alpha\dot\alpha}\lambda^\alpha + \mu_{\dot\alpha})}$$

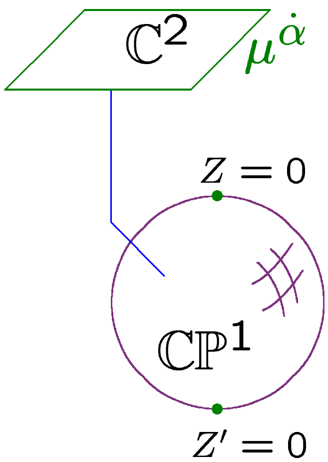
$$\alpha = 1,2, \quad \dot\alpha = \dot 1, \dot 2, \quad x_{\alpha\dot\alpha} = x^\mu \sigma_{\mu\alpha\dot\alpha}.$$

- $\tilde{t} = (\mu, \lambda)$ denotes a $D = 4$ twistor.

- Twistor space is $\mathbb{CP}^3 \setminus \mathbb{CP}^1$ with projective co-ordinates

$$Z^1 = \lambda^1, Z^2 = \lambda^2, Z^3 = \mu^{\dot 1}, Z^4 = \mu^{\dot 2},$$
$$(Z^1, Z^2, Z^3, Z^4) \sim (\zeta Z^1, \zeta Z^2, \zeta Z^3, \zeta Z^4),$$
$$(Z^1, Z^2) \neq (0,0).$$

Picture of twistor space



$\mathbb{CP}^3 \setminus \mathbb{CP}^1$

$Z = \lambda^1/\lambda^2,$

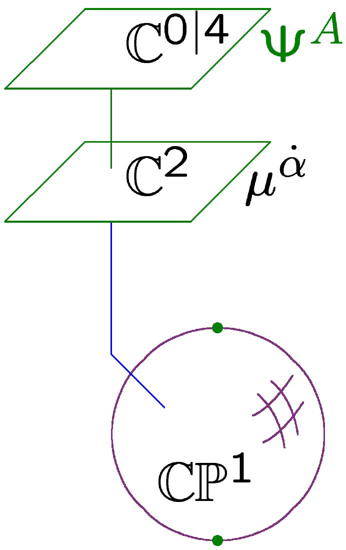
Twistor space is a fibration of $\mathbb{C}^2$ over $\mathbb{CP}^1$.

Two patches:

$\lambda^1 \neq 0$	$\lambda^2 \neq 0$
$Z = \lambda^2/\lambda^1$	$Z' = \lambda^1/\lambda^2 = \frac{1}{Z}$
$X = \mu^{\dot 1}/\lambda^1$	$X' = \mu^{\dot 1}/\lambda^2 = \frac{X}{Z}$
$Y = \mu^{\dot 2}/\lambda^1$	$Y' = \mu^{\dot 2}/\lambda^2 = \frac{Y}{Z}$

Supertwistor space

For $N = 4$ SYM, Witten added four anticommuting coordinates $\psi^1, \ldots, \psi^4$.



$$\mathbb{CP}^{3|4} \setminus \mathbb{CP}^{1|4}$$

Supermanifold

[Sethi, Schwarz, Movshev & Schwarz, Popov & Sämann & Wolf, ...]

Super-twistor space is a fibration of $\mathbb{C}^{2|4}$ over $\mathbb{CP}^1$.

Two patches:

$\lambda^1 \neq 0$	$\lambda^2 \neq 0$
$Z = \lambda^2/\lambda^1$	$Z' = \lambda^1/\lambda^2 = 1/Z$
$X = \mu^{\dot 1}/\lambda^1$	$X' = \mu^{\dot 1}/\lambda^2 = X/Z$
$Y = \mu^{\dot 2}/\lambda^1$	$Y' = \mu^{\dot 2}/\lambda^2 = Y/Z$
$\Theta^A = \psi^A/\lambda^1$	$\Theta'^A = \psi^A/\lambda^2 = \Theta^A/Z$

Superfields

The superfield $\mathcal{A}$ combines the twistor transforms of gluons, fermions and scalars [Witten]:

$$\mathcal{A}(X, Y, Z, \ominus) =$$

$$A + \hat{\hat{\varrho}}_A \ominus^A + \frac{1}{2}\Phi_{AB}\ominus^A\ominus^B$$

$$+\frac{1}{6}\epsilon_{ABCD}\hat{\varrho}^A\ominus^B\ominus^C\ominus^D + \frac{1}{24}G\epsilon_{ABCD}\ominus^A\ominus^B\ominus^C\ominus^D$$

Superfield components

$A(X,Y,Z)$	$(+1)$ helicity gluons $(F=\widetilde{F})$
$\widehat{\widehat{\varrho}}_A(X,Y,Z)$	$(+1/2)$ helicity fermions $(\psi_{\dot{\alpha}A})$
$\phi_{AB}(X,Y,Z)$	(0 helicity) scalars
$\widehat{\varrho}^A(X,Y,Z)$	$(-1/2)$ helicity fermions (ψ_α^A)
$G(X,Y,Z)$	(-1) helicity gluons $(F=-\widetilde{F})$

The contour integrals should be invariant under

$$\mathcal{A} \to \mathcal{A} + (\text{holomorphic at } Z \neq 0) + (\text{holomorphic at } Z \neq \infty)$$

[In other words, $\mathcal{A}$ is an element of sheaf cohomology $H^1(\cdots)$.]

Chiral Fermion Mass Term

Claim #1:

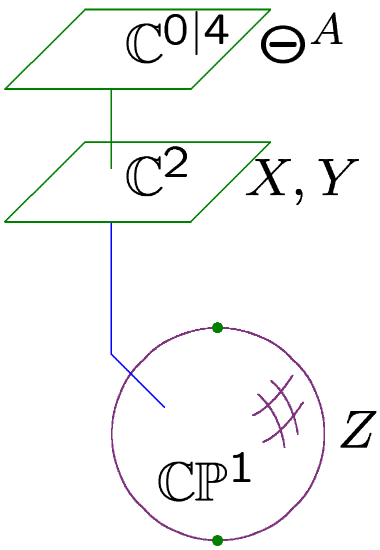
At $O(g^0)$, adding a chiral mass term:

$$\delta\mathcal{L} = \sum_{A,B} M^{AB} \psi_{A\dot{\alpha}} \psi_B^{\dot{\alpha}}$$

is equivalent to a certain super-complex structure deformation of supertwistor space $\mathbb{CP}^{3|4} \setminus \mathbb{CP}^{1|4}$.

Note: The chiral mass term breaks CPT, but all we are doing here is summing Feynman diagrams. We don't care about unitarity . . .

The Θ^3 Supercomplex Structure Deformation



$\mathbb{CP}^{3|4} \setminus \mathbb{CP}^{1|4}$

$\lambda^1 \neq 0$	coordinates for $\lambda^2 \neq 0$
Z	$Z' = \frac{1}{Z}$
X	$X' = \frac{X}{Z}$
Y	$Y' = \frac{Y}{Z}$
Θ^A	$\Theta'^A = \frac{1}{Z}\Theta^A + \frac{1}{6Z^2} M^{AB} \epsilon_{BCDE} \Theta^C \Theta^D \Theta^E$

Wave-functions with the mass term

In momentum space, the free Dirac equation with a chiral mass term is

$$p_{\alpha\dot\alpha}\psi^{\alpha A} = M^{AB}\psi_{\dot\alpha B}, \qquad p_{\alpha\dot\alpha}\psi^{\dot\alpha}_{A} = 0.$$

Like the massless case ($M^{AB} = 0$),

$$p^2 = 0 \qquad \Longrightarrow \qquad p_{\alpha\dot\alpha} = \lambda_\alpha\tilde\lambda_{\dot\alpha}.$$

For the massless case, the general solution is:

$$\psi_{\dot\alpha A} = \tilde\lambda_{\dot\alpha}\tilde\varrho_A(\lambda,\tilde\lambda), \qquad \psi^A_\alpha = \lambda_\alpha\varrho^A(\lambda,\tilde\lambda).$$

For the massive case, the general solution is:

$$\psi_{\dot\alpha A} = \tilde\lambda_{\dot\alpha}\tilde\varrho_A, \qquad \psi^A_\alpha = \lambda_\alpha\varrho^A + M^{AB}\eta_\alpha\tilde\varrho_B,$$

where η_α is some spinor that satisfies

$$\eta_\alpha\lambda^\alpha = 1.$$

(Note that η_α is not globally defined!)

Claim #1 can be justified by analyzing the behavior of the twistor-transform of this solution at $Z = \infty$. [Chiou & OG & Hong & Kim & Mitra]

Holomorphic Curves of Degree $d = 1$

In the undeformed twistor space a holomorphic curve of degree $d = 1$ in $\mathbb{CP}^{3|4}$ is given by a set of linear equations [Witten]

$$X = -x_{1\dot{1}} - x_{2\dot{1}} Z, \qquad Y = -x_{1\dot{2}} - x_{2\dot{2}} Z,$$

$$\Theta^A = -\theta_1^A - \theta_2^A Z,$$

where $x_{\alpha\dot{\alpha}}$ and θ_α^A are moduli.

With the chiral mass term, the last equation has to be replaced with the quadratic expression

$$\boxed{\Theta^A = -\theta_1^A - \theta_2^A Z + M^{AB} \epsilon_{BCDE} \theta_2^C \theta_2^D \theta_2^E Z^2}$$

(In order to have "good" behavior near $Z = \infty$.)

This can be compared with amplitudes . . .

3-Scalar correction

Claim #1':

The above super-complex structure deformation of supertwistor space $\mathbb{CP}^{3|4} \setminus \mathbb{CP}^{1|4}$ is equivalent to adding a chiral mass term and a 3-scalar interaction:

$$\delta\mathcal{L} = \mathrm{tr}\{\sum_{A,B} M^{AB}\psi_{A\dot{\alpha}}\psi_B^{\dot{\alpha}} + gM^{IJK}\phi_I\phi_J\phi_K\}.$$

$$M^{IJK} \equiv \epsilon^{CDEF}\Gamma^{[I}_{AC}\Gamma^{J}_{BD}\Gamma^{K]}_{EF}M^{AB}$$

$$\overline{E}^{AB} = \text{CSUGRA field [Berkovits \& Witten]}$$

$$M^{AB} = \left\langle \overline{E}^{AB} \right\rangle$$

Dimensional Reduction

Can we learn more about mass terms by dimensionally reducing to $D = 3$?

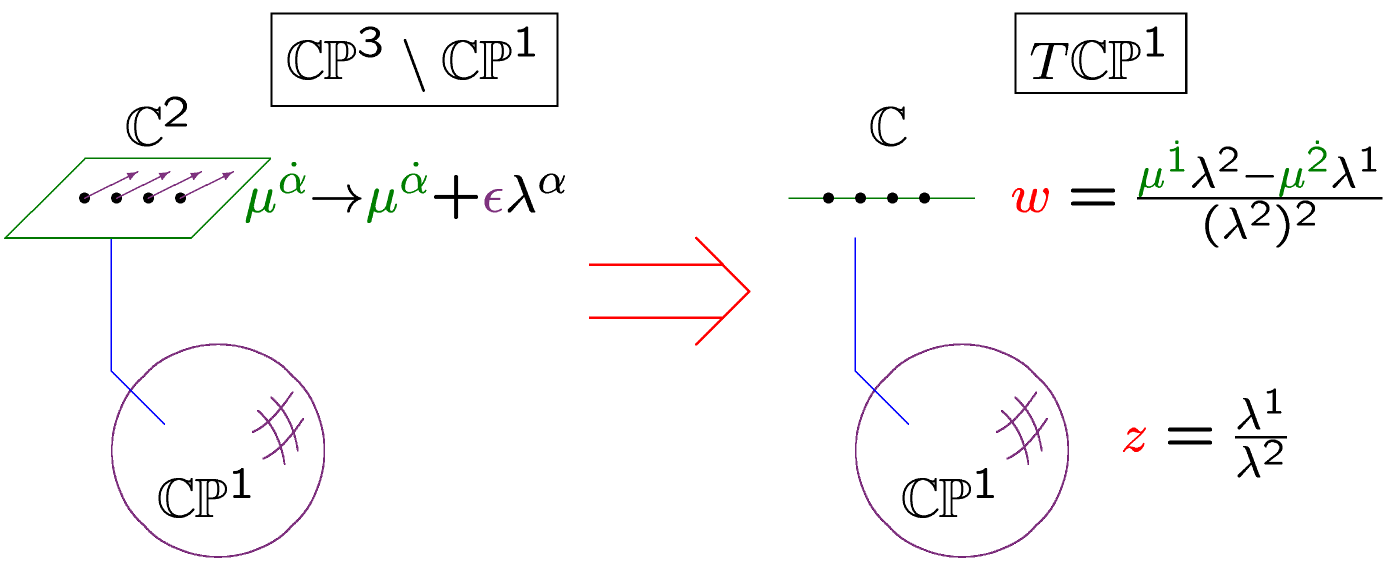
Dimensional Reduction to $D = 3$

We dimensionally reduce to $D = 3$ by gauging the translation generator P_4.

Gauging would make $P_4 = 0$ identically. In an appropriate basis, P_4 acts as

$$\delta\lambda^\alpha = 0, \qquad \delta\mu^{\dot\alpha} = \epsilon\lambda^\alpha.$$

(Note that in $D = 3$ there is no distinction between α and $\dot\alpha$.)



After gauging P_4 we are left with minitwistor space $T\mathbb{CP}^1$. [Hitchin]

Minitwistor space

Minitwistor space is $T\mathbb{CP}^1$ [Hitchin]. It can be parameterized by the P_4-invariant

$$z = \frac{\lambda^1}{\lambda^2}, \qquad w = \frac{\mu^{\dot{1}}\lambda^2 - \mu^{\dot{2}}\lambda^1}{(\lambda^2)^2}$$

For signature $\mathbb{R}^{1,2}$ the minitwistor space is $T\mathbb{RP}^1$ and z, w are real.

The corresponding shock-waves are

$$\phi(x^0, x^1, x^2) = \delta(w + [x^2 - x^0] - 2x^1 z - [x^2 + x^0] z^2)$$

Super-minitwistor space can be covered by two patches with transition relations:

$$z' = \frac{1}{z}, \qquad w' = \frac{1}{z} w, \qquad \theta'^A = \frac{1}{z}\theta^A.$$

Helicity in $D = 3$

The Lagrangian of this $D = 3$ SYM is

$$g_3^2 \mathcal{L} = \mathrm{tr}\Big(\frac{1}{4} F_{ij} F^{ij} + \frac{1}{2} \sum_{i=1}^{7} D_i \Phi^I D^i \Phi^I - \frac{1}{4} \sum_{I,J} [\Phi^I, \Phi^J]^2 + \sum_{a=1}^{8} \chi_\alpha^a \sigma^{i\,\alpha\beta} \partial_i \chi_\beta^a + \sum_{a,b,I} \epsilon^{\alpha\beta} \Gamma^I_{ab} \Phi^I \chi_\alpha^a \chi_\beta^b \Big).$$

Onshell, instead of the gauge field we get two scalars:

$$A_4 = \Phi^7, \qquad F_{ij} = \epsilon_{ijl} \partial_l \Phi^8$$

Helicity $\pm$ refers to onshell states with

$$\Phi^7 = \pm i \Phi^8.$$

Holomorphic curves

At tree-level, Witten's discoveries about $D = 4$ SYM amplitudes and holomorphic curves in twistor space immediately imply similar results for $D = 3$.

For example, MHV amplitudes correspond to quadratic sections of $T\mathbb{CP}^1$:

$$w = -[x^2 - x^0] + 2x^1 z + [x^2 + x^0] z^2.$$

For $D = 3$, there is a correspondence [Hitchin]:

$$\boxed{\begin{array}{c}\text{holomorphic curves}\\ \text{in minitwistor space}\\ T\mathbb{CP}^1\end{array}} \Longleftrightarrow \boxed{\begin{array}{c}\text{Real minimal area}\\ \text{surfaces}\\ \text{in physical space } \mathbb{R}^3\end{array}}$$

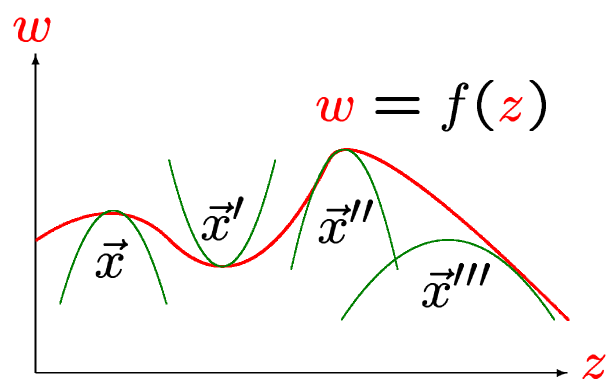
For signature $\mathbb{R}^{1,2}$, this correspondence translates to an amusing physical interpretation for the holomorphic curves.

Algebraic curves in $T\mathbb{RP}^1$ and filaments in $\mathbb{R}^{1,2}$

$$0 = \sum_{r,s} z^r w^s \qquad \text{Expand near } (w_0, z_0) \Longrightarrow$$

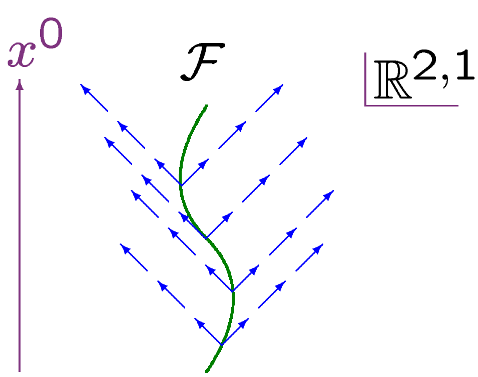
$$w = w_0 + a_1(z - z_0) + a_2(z - z_0)^2 + O(z - z_0)^3$$

We approximate the algebraic curve locally by parabolas. Each parabola corresponds to an MHV curve.



Each parabola therefore corresponds to a point $\vec{x}$ ($\vec{x}', \vec{x}'', \ldots$) in physical space $\mathbb{R}^{1,2}$. The collection of the points $\vec{x}, \vec{x}', \vec{x}'', \ldots$ forms a filament $\mathcal{F}$.

The filament is a null worldline in $\mathbb{R}^{1,2}$!



The outgoing waves of the scattering process can now be described as a physical disturbance that is emanating from the filament $\mathcal{F}$.

Twisted Dimensional Reduction

We can get $D = 3$ mass terms by gauging a linear combination of translation and $SU(4)$ R-symmetry:

$$P_4 - M^A{}_B R_A{}^B = 0.$$

$R_A{}^B$ is the R-symmetry charge. E.g.,

$$[R_A{}^B, \psi^C] = \delta_A^C \psi^B.$$

For example, Dirac's equation becomes

$$0 = \sum_{\mu=1}^{4} \Gamma^\mu \partial_\mu \psi^A = \sum_{i=1}^{3} \Gamma^i \partial_i \psi^A + i M^A{}_B \psi^B.$$

There is also a mass term for the scalars

$$0 = \partial_i \phi^{[AB]} \partial^i \phi_{[AB]} + M^A{}_C M^B{}_D \phi_{[AB]} \phi^{[CD]},$$

where

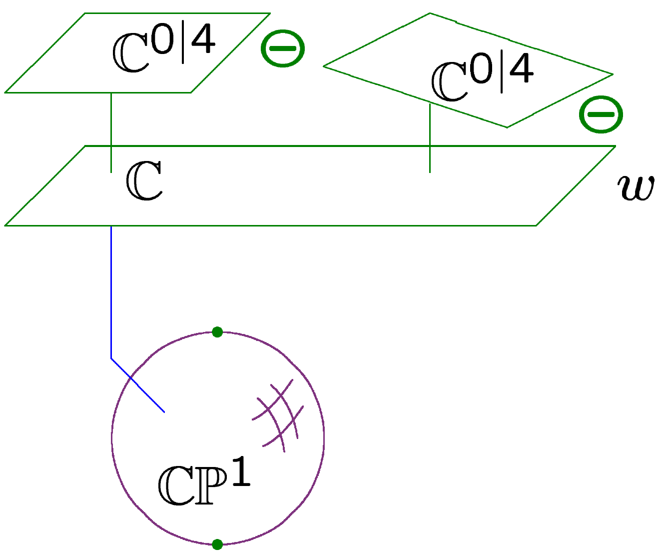
$$\phi_{AB} \equiv \tfrac{1}{2} \epsilon_{ABCD} \phi^{CD}$$

Repeating the steps as before, we get instead of minitwistor superspace . . .

$D = 3$ Massive Super-mini-twistor space

Repeating the steps as before, we get the super-mini-twistor target space for the massive $D = 3$ SYM in the form

$$\boxed{Z' = \frac{1}{Z}, \qquad W' = \frac{W}{Z}, \qquad \Theta' = \frac{1}{Z}\exp\Big(\frac{W}{Z}M\Big)\Theta}$$



The four anticommuting Θ^A directions are fibered in a nontrivial way over the W-plane.

How is this related to direct dimensional reduction of the mass term deformation

$$\boxed{\Theta'^A = \frac{1}{Z}\Theta^A + \frac{1}{6Z^2}M^{AB}\epsilon_{BCDE}\Theta^C\Theta^D\Theta^E}$$

that we found previously?

$D = 3$ Infinitesimal Mass Terms

For infinitesimal mass terms in $D = 3$ we get the following complex structure deformations

$$\delta M^A{}_B \sigma^4_{\alpha\dot{\alpha}} \psi^{\dot{\alpha}}_A \psi^{B\alpha} \Longrightarrow \delta\Theta'^A = \delta M^A{}_B \frac{W}{Z^2} \Theta^B,$$

$$\delta M^{AB} \psi^{\dot{\alpha}}_A \psi_{B\dot{\alpha}} \Longrightarrow \delta\Theta'^A = \delta M^{AB} \frac{1}{6Z^2} \epsilon_{BCDE} \Theta^C \Theta^D \Theta^E$$

The vector fields on the RHS are the only translationally invariant (in cohomology) $\delta\Theta'$ deformations (unless we allow anticommuting parameters).

The translation generators act as

$$P_1 = iZ \frac{\partial}{\partial W},$$

$$P_+ := P_2 + iP_3 = -i \frac{\partial}{\partial W},$$

$$P_- := P_2 - iP_3 = iZ^2 \frac{\partial}{\partial W}.$$

R-symmetry [Spin(7) in $D = 3$] should transform one mass term to the other. How does Spin(7) act?

Summary

- In $D = 3$ we found that

$$\boxed{Z' = \frac{1}{Z}, \qquad W' = \frac{W}{Z}, \qquad \Theta' = \frac{1}{Z}\exp\left(\frac{W}{Z}M\right)\Theta}$$

  corresponds to a mass term.

- In $D = 4$ we found that

$$\boxed{\Theta'^A = \frac{1}{Z}\Theta^A + \frac{1}{6Z^2}M^{AB}\epsilon_{BCDE}\Theta^C\Theta^D\Theta^E}$$

  corresponds to a chiral mass term.

- In $D = 3$ holomorphic curves in twistor space correspond to filaments in spacetime from which the scattered wave-functions originate.

Open issues

- $D = 3$ at 1-loop?
- The limit $M \to \infty$?
- The Spin(7) R-symmetry ...
- Mirror symmetry ...